Chapter II
The Category of Simplicial Complexes

II.1 Euclidean Simplicial Complexes

Let us recall that a subset $C \subset \mathbb{R}^n$ is *convex* if $x, y \in C, t \in [0,1] \implies tx + (1-t)y \in C$. The *convex hull* of a subset $X \subset \mathbb{R}^n$ is the smallest convex subset of $\mathbb{R}^n$, which contains X. We say that $d+1$ points $x_0, x_1, \ldots, x_d$ belonging to the Euclidean space $\mathbb{R}^n$ are *linearly independent* (from the affine point of view) if the vectors $x_1 - x_0$, $x_2 - x_0$, $\ldots$, $x_d - x_0$ are linearly independent. A vector $x - x_0$ of the vector space generated by these vectors can be written as a sum $x - x_0 = \sum_{i=1}^{d} r_i(x_i - x_0)$ with real coefficients r_i; notice that if we write x as $x = \sum_{i=0}^{d} \alpha_i x_i$, then $\sum_{i=0}^{d} \alpha_i = 1$. If $\{x_0, \ldots, x_d\} \in X$ are affinely independent, the convex hull of X is said to be an (Euclidean) *simplex* of dimension d contained in $\mathbb{R}^n$; its points x can be written in a unique fashion as linear combinations

$$x = \sum_{i=1}^{d} \lambda_i x_i,$$

with real coefficients λ_i. The coefficients λ_i are called *barycentric coordinates* of x; they are nonnegative real numbers and satisfy the equality $\sum_{i=0}^{d} \lambda_i = 1$. The points x_i are the *vertices* of the simplex. The *standard n-simplex* is the simplex obtained by taking the convex hull of the $n+1$ points of the standard basis of $\mathbb{R}^{n+1}$ (see Figs. II.1 and II.2 for dimensions $n = 1$ and $n = 2$, respectively).

The *faces* of a simplex $s \subset \mathbb{R}^n$ are the convex hulls of the subsets of its vertices; the faces which do not coincide with s are the *proper faces*. We can define the *interior* of a simplex s as the set of all points of s with positive barycentric coordinates $\lambda_i > 0$. We indicate the interior of s with $\mathring{s}$. If the dimension of s is at least 1, $\mathring{s}$ coincides with the topological interior. At any rate, it is not hard to prove that we obtain the interior of a simplex by removing all of its proper faces.

An Euclidean *simplicial complex* is a finite family of simplexes of an Euclidean space $\mathbb{R}^n$, which satisfies the following properties:

D.L. Ferrario and R.A. Piccinini, *Simplicial Structures in Topology*,
CMS Books in Mathematics, DOI 10.1007/978-1-4419-7236-1_II,

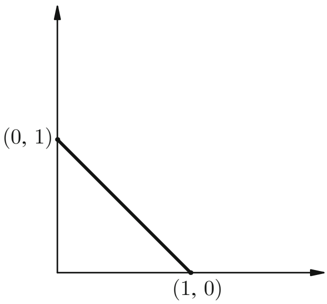

Fig. II.1

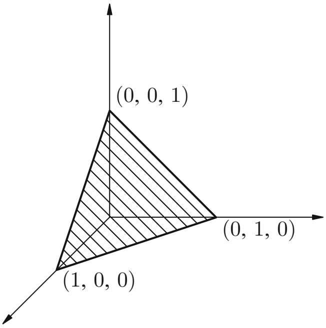

Fig. II.2

S1 If $s \in K$, then every face of s is in K.
S2 If s_1 and s_2 are simplexes of K with nondisjoint interiors $\mathring{s}_1 \cap \mathring{s}_2 \neq \emptyset$, then $s_1 = s_2$.

The *dimension* of K is the maximal dimension of its simplexes.[1] Figure II.3 repre-

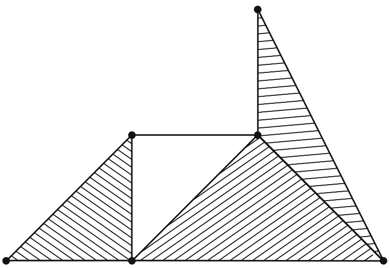

Fig. II.3

sents a two-dimensional simplicial complex of $\mathbb{R}^2$; Fig. II.4 is a set of simplexes, which is *not* a simplicial complex.

[1] It is possible to define Euclidean complexes with infinitely many simplexes, provided we add the *local finiteness* property that is to say, we ask that each point of a simplex has a neighborhood, which intersects only finitely many simplexes of K. We do this so that the topology of the (infinite) Euclidean complex K coincides with the topology of the geometric realization $|\widehat{K}|$ (we are referring to the topology defined by Remark (II.2.13)) of the abstract simplicial complex $|\widehat{K}|$ associated in a natural fashion to K (we shall give the definition of abstract simplicial complex in a short while).

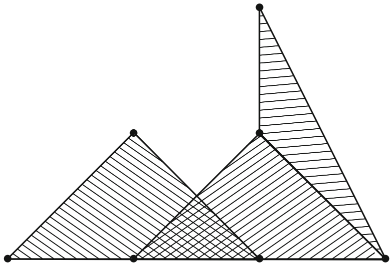

Fig. II.4

(II.1.1) Example (Euclidean polyhedra)**.** 1. Every simplex of $\mathbb{R}^n$ together with all its faces is a simplicial complex.

2. The set of all proper faces of a d-dimensional simplex in $\mathbb{R}^n$ is a $(d-1)$-dimensional simplicial complex.
3. The set of all closed intervals $[1/n, 1/(n+1)]$, with $n \in \mathbb{N}$, is a simplicial complex (with infinitely many simplexes) of $\mathbb{R}$.
4. Let P_m be the regular polygonal line contained in $\mathbb{C} \cong \mathbb{R}^2$, whose vertices are the mth-roots of the unity $\{z \in \mathbb{C} \,|\, z^m = 1\}$. The corresponding simplicial complex is homeomorphic to the circle S^1 and is depicted in Fig. II.5.

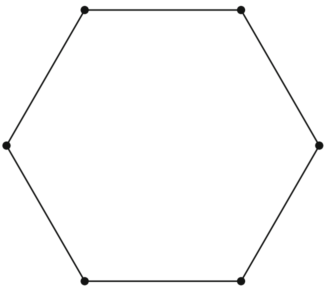

Fig. II.5

5. The Platonic solids can be subdivided by triangles; they give rise to simplicial complexes of $\mathbb{R}^3$. An example is given by the icosahedron of Fig. II.6.

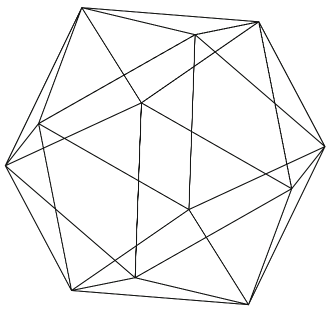

Fig. II.6

II.2 Abstract Simplicial Complexes

In this section, we shall give the definition of the category **Csim** of simplicial complexes and simplicial maps; furthermore, we shall define two important functors with domain **Csim**, namely, the geometric realization functor and the homology functor.

An *(abstract) simplicial complex* is a pair $K = (X, \Phi)$ given by a *finite* set X and a set of nonempty subsets of X such that:

K1 $(\forall x \in X)\ ,\ \{x\} \in \Phi$,
K2 $(\forall \sigma \in \Phi)(\forall \sigma' \subset \sigma\ ,\ \sigma' \neq \emptyset)\ ,\ \sigma' \in \Phi$.

The elements of X are the *vertices* of K. The elements of Φ are the *simplexes* of K. If σ is a simplex of K, every non-empty $\sigma' \subset \sigma$ is a *face* of σ. According to condition K2, we can say that all faces of a simplex are simplexes. A simplex σ with $n+1$ elements ($n \geq 0$) is an *n-simplex* (we also say that σ is a simplex of *dimension n*); we adopt the notation $\dim \sigma = n$. It follows that the 0-simplexes are *vertices*. The dimension of K is the maximal dimension of its simplexes; if the dimensions of all simplexes of K have a maximum n, we say that K has *dimension n* or that K is *n-dimensional*.

(II.2.1) Remark. We explicitly observe that in this book all simplicial complexes have a finite number of vertices.

Before we present some examples and constructions with simplicial complexes, we give a definition: a simplicial complex $L = (Y, \Psi)$ is a *subcomplex* of $K = (X, \Phi)$ if $Y \subset X$ and $\Psi \subset \Phi$.

(II.2.2) Remark. Let $K_0 = (X_0, \Phi_0)$ and $K_1 = (X_1, \Phi_1)$ be subcomplexes of a simplicial complex K; we observe that the union $K_0 \cup K_1 = (X_0 \cup X_1, \Phi_0 \cup \Phi_1)$ and the intersection $K_0 \cap K_1 = (X_0 \cap X_1, \Phi_0 \cap \Phi_1)$ (with $X_0 \cap X_1 \neq \emptyset$) are subcomplexes of K. In particular, the union of two *disjoint* simplicial complexes K_0 and K_1 (that is to say, such that $X_0 \cap X_1 = \emptyset$) is a simplicial complex.

Let us now give some examples.

1. Let X be a finite set and let $\wp(X) = 2^X$ be the set of all subsets of X; clearly, the pair $K = (X, \wp(X) \setminus \emptyset)$ is a simplicial complex.
2. The set of all simplexes of an Euclidean simplicial complex is an abstract simplicial complex if we forget the fact that its vertices are points of $\mathbb{R}^n$. The set X is the set of all vertices, while Φ is the set of simplexes. Thus, the examples of Euclidean polyhedra on p. 45 are examples of abstract simplicial complexes.
3. Let Γ be a graph (that is to say, a set of vertices X and a symmetric subset Φ of $X \times X$, called *set of edges*). It is not hard to prove that (X, Φ) is a simplicial complex if we assume that $\sigma \in \Phi \subset 2^X$ whenever σ is a set with just one element or is the set of the two vertices at the ends of an edge.

(II.2.3) Definition (generated complex)**.** Let $K = (X, \Phi)$ be a simplicial complex; for every simplex $\sigma \in \Phi$, the pair

$$\overline{\sigma} = (\sigma, \wp(\sigma) \setminus \emptyset)$$

is a simplicial complex; $\overline{\sigma}$ is the simplicial complex *generated* by σ (sometimes also called *closure* of σ). More generally, let B be a set of simplexes of K, that is to say, $B \subset \Phi$; then

$$\overline{B} = \bigcup_{\sigma \in B} \overline{\sigma}$$

is the simplicial complex *generated* by all the simplexes of the set B. Observe that $\overline{B}$ is a subcomplex of K.

(II.2.4) Definition (boundary of a simplex)**.** For every simplex σ of a simplicial complex,

$$\dot{\sigma} = (\sigma, \wp(\sigma) \setminus \{\emptyset, \sigma\})$$

is a simplicial complex, called *boundary* of σ. By an abuse of notation, we write

$$\overline{\sigma} = \dot{\sigma} \cup \sigma.$$

(II.2.5) Definition (join and suspension)**.** Given two simplicial complexes $K = (X, \Phi)$ and $L = (Y, \Psi)$, the *join* of K and L is the simplicial complex $K * L$ whose vertices are all the elements of the set $X \cup Y$, and whose simplexes are the elements of the sets Φ, Ψ and of the set

$$\Phi * \Psi = \{\{x_0, \ldots, x_n, y_0, \ldots, y_m\} \mid \{x_0, \ldots, x_n\} \in \Phi\ , \{y_0, \ldots, y_m\} \in \Psi\}.$$

In other words, a nonempty subset $\{x_0, \ldots, x_n, y_0, \ldots, y_m\}$ of $X \cup Y$ is a simplex of $K * L$ if and only if $\{x_0, \ldots, x_n\} \in \Phi \cup \{\emptyset\}$ and $\{y_0, \ldots, y_m\} \in \Psi \cup \{\emptyset\}$. In particular, if $L = (Y, \Psi)$ is the simplicial complex defined by a unique point y, $K * y = Ky$ is the *cone* (sometimes called *abstract cone*) of K with vertex y (of course, we can also define the cone yK). An n-simplex with $n \geq 1$ can be interpreted as the cone of any of its faces (of dimension $n - 1$).

If L is the simplicial complex determined by exactly two points x and y, that is to say,

$$L = (Y, \Psi) \text{ with } Y = \{x, y\}\ ,\ \Psi = \{\{x\}, \{y\}\}\ ,$$

the join $K * L = \Sigma K$ is called *suspension* of K. Observe that ΣK can be viewed as the union of the cones $K * x$ and $K * y$.

The category **Csim** of *simplicial complexes* is the category whose objects are all simplicial complexes, and whose morphisms $f : K = (X, \Phi) \to L = (Y, \Psi)$ are the functions (between sets) $f : X \to Y$ such that

$$(\forall \sigma = \{x_0, x_1, \ldots, x_n\} \in \Phi)\, ,\ f(\sigma) = \{f(x_0), f(x_1), \ldots, f(x_n)\} \in \Psi.$$

A morphism $f \in \mathbf{Csim}(K, L)$ is a *simplicial function* from K to L.

II.2.1 The Geometric Realization Functor

For a given simplicial complex $K = (X, \Phi)$, let $V(K)$ be the set of all functions $p\colon X \to \mathbb{R}_{\geq 0}$ (nonnegative real numbers); we define the *support* of an arbitrary $p \in V(K)$ to be the finite set

$$s(p) = \{x \in X \mid p(x) > 0\}.$$

Let $|K|$ be the set defined as follows:

$$|K| = \{p \in V(K) \mid s(p) \in \Phi \text{ and } \sum_{x\in s(p)} p(x) = 1\}.$$

We now define the function

$$d\colon\ |K| \times |K| \longrightarrow \mathbb{R}_{\geq 0}$$

which takes any pair $(p,q) \in |K| \times |K|$ into the real number

$$d(p,q) = \sqrt{\sum_{x\in X} (p(x) - q(x))^2}.$$

This function is a metric on K (verify the conditions defining a metric given in Sect. I.1.5); hence, it defines a (metric) topology on $|K|$. The metric space $|K|$ is the *geometric realization* of K. Observe that $|K|$ is a *bounded* space, in the sense that $(\forall p,q \in |K|)$, $d(p,q) \leq \sqrt{2}$. Moreover, $|K|$ is a Hausdorff space.

We can write the elements of K as finite linear combinations. In fact, for each vertex x of K, with a slight abuse of language, let us denote with x the function of $V(K)$, with value 1 at the vertex x and 0 at any other vertex; in a more formal fashion,

$$(\forall y \in X)\ x(y) = \begin{cases} 0 \text{ if } y \neq x \\ 1 \text{ if } y = x \end{cases}$$

(in other words, we identify the vertex x with the corresponding real function of $V(K)$, whose support coincides with the set $\{x\}$). Hence if $s(p) = \{x_0, x_1, \ldots, x_n\}$ is the support of $p \in |K|$, and assuming that $p(x_i) = \alpha_i$, $i = 0, 1, \ldots, n$, we can write p as

$$p = \sum_{i=0}^{n} \alpha_i x_i.$$

The real numbers α_i , $i = 0, \ldots, n$, are the *barycentric coordinates* of p (in agreement with the barycentric coordinates defined by $n+1$ independent points of an Euclidean space).

(II.2.6) Remark. Because K has a finite number of vertices, say n, we can embed the set of vertices X in the Euclidean space $\mathbb{R}^n$, so that the images of the elements of X coincide with the vectors of the standard basis. Then we can take the convex hulls in $\mathbb{R}^n$ of the vectors corresponding to the simplexes of K, to obtain an Euclidean simplicial complex $K' \subset \mathbb{R}^n$ associated with K. We shall see in a short while that K' is isomorphic to the geometric realization $|K|$ (actually, there exists an isometry between these two metric spaces; furthermore, the set of all functions $X \to \mathbb{R}_{\geq 0}$ coincides with the positive quadrant of $\mathbb{R}^n$).

The following statement holds true: two points $p, q \in |K|$ coincide if and only if they have the same barycentric coordinates.

The *geometric realization functor*

$$| \ |\colon \ \mathbf{Csim} \longrightarrow \mathbf{Top}$$

is defined over an object $K \in \mathbf{Csim}$ as the geometric realization $|K|$, and over a morphism $f \in \mathbf{Csim}(K, L)$ as

$$|f|\colon \ |K| \to |L|\,,\ |f|(\sum \alpha_i x_i) = \sum \alpha_i f(x_i)\,.$$

To prove that $| \ |$ is indeed a functor, we need the following result.

(II.2.7) Theorem. *The function $|f|$ induced from a simplicial function $f\colon \ K \to L$ is continuous.*

Proof. It is enough to prove that, for every $p \in |K|$, there exists a constant $c(p) > 0$ which depends on p and such that, for every $q \in |K|$, $d(|f|(p), |f|(q)) \leq c(p)d(p, q)$.

Assume that

$$s(p) = \{x_0, \ldots, x_n\} \ \text{ and } \ s(q) = \{y_0, \ldots, y_m\}$$

and also that $p(x_i) = \alpha_i$ for $i = 0, \ldots, n$, and $q(y_j) = \beta_j$ for $j = 0, \ldots, m$. We consider three cases.

Case 1: $s(p) \cap s(q) = \emptyset$ - In this situation

$$d(p, q) = \sqrt{\sum_{i=0}^{n} \alpha_i^2 + \sum_{j=0}^{m} \beta_j^2} \geq \sqrt{\sum_{i=0}^{n} \alpha_i^2}\ ;$$

because $\sum_{i=0}^{n} \alpha_i = 1$, $\sum_{i=0}^{n} \alpha_i^2$ has its minimum value only when $\alpha_i = 1/(n+1)$, for every $i = 0, \ldots, n$. It follows that $d(p, q) \geq 1/\sqrt{n+1}$ and

$$\frac{d(|f|(p), |f|(q))}{d(p, q)} \leq \frac{\sqrt{2}}{1/\sqrt{n+1}}$$

(recall that $d(|f|(p), |f|(q)) \leq \sqrt{2}$); so,

$$d(|f|(p), |f|(q)) \leq \sqrt{2(n+1)}\, d(p, q);$$

thus, we define $c(p) = \sqrt{2(n+1)}$.

Case 2: $s(p) \cap s(q) \neq \emptyset$, but $s(p) \not\subset s(q)$ and $s(q) \not\subset s(p)$ –

Let us rewrite the indices of the elements of $s(p)$ and $s(q)$ to have the following common elements:

$$x_r = y_0, x_{r+1} = y_1, \ldots, x_n = y_{n-r}.$$

Notice that the set $s(p) \cup s(q)$ has exactly $m+r+1$ common elements. Now consider the elements

$$z_i = \begin{cases} x_i, & 0 \le i \le r-1 \\ x_i = y_{i-r}, & r \le i \le n \\ y_{i-r}, & n+1 \le i \le m+r \end{cases}$$

together with the real numbers

$$\gamma_i = \begin{cases} -\alpha_i, & 0 \le i \le r-1 \\ -\alpha_i + \beta_{i-r}, & r \le i \le n \\ \beta_{i-r}, & n+1 \le i \le m+r\,. \end{cases}$$

Notice that $\gamma_i < 0$ for $i = 0, \ldots, r-1$ and $\gamma_i > 0$ for $i = n+1, \ldots, m+r$, because $\alpha_i > 0$ for every $i = 0, 1, \ldots, n$ and $\beta_i > 0$ for $i = 1, \ldots, m$. Let us order the numbers γ_i in such a way that $\gamma_0 \le \gamma_1 \le \ldots \le \gamma_{m+r}$ (if necessary, we make a permutation of the indices). Let l be the largest index for which $\gamma_l < 0$ (because of the assumptions we made, such a set of indices cannot be empty - thus l exists - nor can it be the set of all indices - thus $r \le l \le n$); moreover, the vertices (viewed as functions) $z_0, z_1, \ldots, z_l$ are summands of p (the numbers γ_i are negative), while the vertices $z_{l+1}, \ldots, z_{m+r}$ are part of q (the corresponding numbers γ_i are non-negative). At this point, take $\lambda = \sum_{i=0}^{l} \gamma_i < 0$ and the two finite successions of real positive numbers

$$\{\frac{\gamma_0}{\lambda}, \ldots, \frac{\gamma_l}{\lambda}\} \text{ and } \{\frac{\gamma_{l+1}}{-\lambda}, \ldots, \frac{\gamma_{m+r}}{-\lambda}\}.$$

The elements

$$p' = \sum_{i=0}^{l} \frac{\gamma_i}{\lambda} z_i \text{ and } q' = \sum_{i=l+1}^{m+r} \frac{\gamma_i}{-\lambda} z_i$$

are in $|K|$ because

$$\sum_{i=0}^{l} \frac{\gamma_i}{\lambda} = \sum_{i=l+1}^{m+r} \frac{\gamma_i}{-\lambda} = 1;$$

from what we proved above, it follows that $s(p') \subset s(p)$ and $s(q') \subset s(q)$. But $s(p') \cap s(q') = \emptyset$ and so, by *Case 1*,

$$d(|f|(p'), |f|(q')) \le \sqrt{2(l+1)}\, d(p', q').$$

The equalities

$$d(p', q') = \frac{1}{-\lambda} d(p, q),$$

$$d(|f|(p'), |f|(q')) = \frac{1}{-\lambda} d(|f|(p), |f|(q)),$$

and the fact that $\sqrt{2(l+1)} \le \sqrt{2(n+1)}$ allow us to conclude that

$$d(|f|(p),|f|(q)) \leq \sqrt{2(n+1)}d(p,q).$$

Case 3: Let us assume that $s(p) \subset s(q)$. Rewrite the indices of the elements of $s(p)$ and $s(q)$ in such a way that, $x_i = y_i$ for every $i = 0,\dots,n$. Similar to the previous case, we consider the elements

$$z_i = \begin{cases} x_i = y_i, & 0 \leq i \leq n \\ y_j, & n+1 \leq j \leq m \end{cases}$$

and the real numbers

$$\gamma_i = \begin{cases} -\alpha_i + \beta_i, & 0 \leq i \leq n \\ \beta_j, & n+1 \leq j \leq m. \end{cases}$$

If $-\alpha_i + \beta_i \geq 0$ for every $i = 0,\dots,n$, then $s(p) = s(q)$ and $p = q$, because $\sum_{i=0}^n \alpha_i = 1$. Hence, there exists a number $0 \leq i \leq n$ such that $-\alpha_i + \beta_i < 0$. At this point, we argue as in the previous case. If $s(q) \subset s(p)$, we use an analogous procedure. ■

In particular, the following result holds true:

(II.2.8) Theorem. *Any piecewise linear function (the simplicial realization of a simplicial function)*

$$F\colon\ |K| \to |L|\ ,\ \ F\left(\sum_{i=0}^n \alpha_i x_i\right) = \sum_{i=0}^n \alpha_i F(x_i)$$

is continuous.

Hence $|\ \ |$ is a functor.

We now investigate some of the properties of the geometric realization of a simplicial complex. Recall that it is possible to characterize a convex set X of an Euclidean space as follows: for every $p,q \in X$, the segment $[p,q]$, with end-points p and q, is contained in X. As we are going to see in the next theorem, this convexity property is valid for the geometric realization of the complex $\overline{\sigma}$ (called *geometric simplex*), for every simplex σ of a simplicial complex K.

(II.2.9) Theorem. *Let $K = (X,\Phi)$ be a simplicial complex. The following results hold true:*

(i) The geometric realization $\overline{\sigma}$ of any simplex $\sigma \in \Phi$ is convex.
(ii) For every two simplexes $\sigma,\tau \in \Phi$ we have

$$|\overline{\sigma}| \cap |\overline{\tau}| = |\overline{\sigma \cap \tau}|.$$

(iii) For every $\sigma \in \Phi$, $\overline{\sigma}$ is compact.

Proof. (i) Assume that $\sigma = \{x_0,\dots,x_n\}$ and let p,q be arbitrary points of $|\overline{\sigma}|$; suppose that $p = \sum_{i=0}^n \alpha_i x_i$ and $q = \sum_{i=0}^n \beta_i x_i$. The segment $[p,q]$ is the set of all points $r = tp + (1-t)q$, for every $t \in [0,1]$. Then

$$r = tp + (1-t)q = \sum_{i=0}^{n} (t\alpha_i + (1-t)\beta_i)x_i$$

with $\sum_{i=0}^{n}(t\alpha_i + (1-t)\beta_i) = 1$ and so, $r \in |\overline{\sigma}|$.
(ii) Let us first observe that if $p \in |\overline{\sigma}|$, then $s(p) \subset \sigma$. Now if $p \in |\overline{\sigma}| \bigcap |\overline{\tau}|$, $s(p) \subset \sigma \cap \tau$, and thus $p \in |\overline{s(p)}| \subset |\overline{\sigma \cap \tau}|$. Conversely, if $p \in |\overline{\sigma \cap \tau}|$ then $p \in |\overline{s(p)}| \subset |\overline{\sigma}|$, $p \in |\overline{s(p)}| \subset |\overline{\tau}|$, and therefore $p \in |\overline{\sigma}| \bigcap |\overline{\tau}|$.
(iii) Take the standard n-simplex

$$\Delta^n = \{(z_0, \ldots, z_n) \in \mathbb{R}^{n+1} | 0 \leq z_i \leq 1, \sum_i z_i = 1\}$$

endowed with a system of barycentric coordinates with respect to the vertices

$$e_0 = (1,0,\ldots,0), \ldots e_n = (0,0,\ldots,1);$$

we can write the elements of Δ^n as linear combinations with nonnegative real coefficients $\sum_{i=0}^{n} \alpha_i e_i$ where $\sum_{i=0}^{n} \alpha_i = 1$. Furthermore, we observe that Δ^n is compact as a bounded and closed subset of $\mathbb{R}^{n+1}$ (see Theorem (I.1.36)). Let $f\colon\ \Delta^n \to |\overline{\sigma}|$ be the function taking any $p = \sum_{i=0}^{n} \alpha_i e_i \in \Delta^n$ to the point $f(p) = \sum_{i=0}^{n} \alpha_i x_i$. This function is bijective, continuous, and takes a compact space to a Hausdorff space; hence, f is a homeomorphism (see Theorem (I.1.27)). It follows that $|\overline{\sigma}|$ is compact. ■

As we have observed before, the geometric realization $|K|$ can be viewed as a subspace of $\mathbb{R}^n$, where n is the number of vertices of K. Thus, it is possible to consider an affine structure on the ambient space $\mathbb{R}^n$, and again analyze the convexity of the various parts of K and the linear combinations of elements with barycentric coordinates. For every $p \in |K|$, let $B(p)$ be the set of all $\sigma \in \Phi$ such that $p \in |\overline{\sigma}|$; now take the space

$$D(p) = \bigcup_{\sigma \in B(p)} |\overline{\sigma}|\ .$$

The *boundary* $S(p)$ of $D(p)$ is the union of the geometric realizations of the complexes generated by the faces $\tau \subset \sigma$, with $\sigma \in B(p)$ and $p \notin |\overline{\tau}|$. Intuitively, $D(p)$ is the "disk" defined by all geometric simplexes, which contain p and $S(p)$ is its bounding "sphere". Observe that $D(p)$ and $S(p)$ are closed subsets of $|K|$; finally, leaving out the geometric realization, $D(p)$ and $S(p)$ are subcomplexes of K.

(II.2.10) Theorem. *Let K be a simplicial complex; the following properties are valid.*

(i) For every $p \in |K|$, $D(p)$ is compact.
(ii) For every $q \in D(p) \setminus \{p\}$ and every $t \in I$, the point $r = (1-t)p + tq$ belongs to $D(p)$.
(iii) Every ray in $D(p)$ with origin p intersects $S(p)$ at a unique point.

Proof. (i): The compactness of $D(p)$ follows from Theorem (II.2.9), (iii).
(ii): Because $q \in D(p) \setminus \{p\}$, there exists a simplex $\sigma \in B(p)$ such that $q \in |\overline{\sigma}|$, a convex space; it follows that the segment $[p,q]$ is entirely contained in $|\overline{\sigma}| \subset D(p)$.

(iii): Let ℓ be a ray with origin p, and let q be the point of ℓ determined by the condition

$$d(p,q) = \sup\{d(p,q') \mid q' \in \ell \cap D(p)\}.$$

Then, $|s(q)| \in S(p)$; otherwise, we could extend ℓ in $D(p)$ beyond q and thus we would have $q \in S(p)$. On the other hand, because $s(q)$ is a face of a simplex containing p, the vertices of $s(p)$ and $s(q)$ define a simplex of which $s(p)$ is a face (in fact, $s(p) \cup s(q)$ is a simplex of the simplicial complex $D(p)$). The points of ℓ beyond q cannot be in $S(p)$ and the open segment (p,q) is contained in $D(p) \smallsetminus S(p)$. Hence, ℓ intersects $S(p)$ in one point only. ■

Notice that $D(p)$ is not necessarily convex; at any rate, as we have seen in part (ii) of the previous theorem, $D(p)$ is endowed with a certain kind of convexity in the sense that, for every $q \in D(p)$, the segment $[p,q]$ is entirely contained in $D(p)$. We say that $D(p)$ is *p-convex* (star convex). Theorem (II.2.10) allows us to define a map

$$\pi_p \colon D(p) \smallsetminus \{p\} \to S(p) \ , \ q \mapsto \ell_{p,q} \cap S(p)$$

where $\ell_{p,q}$ is the ray with origin p and containing q; the function π_p is the *radial projection with center p* from $D(p)$ onto $S(p)$. Let $i \colon S(p) \to D(p) \smallsetminus \{p\}$ be the inclusion map; then $\pi_p i = 1_{S(p)}$, and $i\pi_p$ is homotopic to the identity map of $D(p) \smallsetminus \{p\}$ onto itself with homotopy given by the map

$$H \colon (D(p) \smallsetminus \{p\}) \times I \to D(p) \smallsetminus \{p\} \ , \ (q,t) \mapsto (1-t)q + t\pi_p(q).$$

Hence, $S(p)$ is a deformation retract of $D(p) \smallsetminus \{p\}$ (see Exercise 2, Sect. I.2). This shows another similarity between the spaces $D(p)$, $S(p)$ and, respectively, the n-dimensional Euclidean disk and its boundary.

The next result (cf. [24]) will be used only when studying triangulable manifolds (Sect. V.1); the reader could thus leave it for later on.

(II.2.11) Theorem. *Let $f \colon |K| \to |L|$ be a homeomorphism. Then, for every $p \in |K|$, $S(p)$ and $S(f(p))$ are of the same homotopy type.*

Proof. Assume that $s(f(p)) = \{y_0, \dots, y_n\}$ and let $U = |\overline{s(f(p))}| \smallsetminus |\dot{s}(f(p))|$ be the interior of $|\overline{s(f(p))}|$, that is to say, the set of all $q \in |L|$ such that $q(y_i) > 0$, for every $i = 0, \dots, n$. Notice that U is an open set of $D(f(p))$; moreover, $f^{-1}(U)$ is an open set of $|K|$ containing p. The bounded, compact set $D(p)$ can be shrunk at will: in fact, for any real number $0 < \lambda \leq 1$ we define the *compression* $\lambda D(p)$ as the set of all points $r = (1-\lambda)p + \lambda q$, for every $q \in D(p)$; observe that $\lambda D(p)$ is a closed subset of $|K|$, and is homeomorphic to $D(p)$. Let $\lambda \in (0,1]$ be such that

$$p \in \lambda D(p) \subset f^{-1}(U);$$

then

$$f(p) \in f(\lambda D(p)) \subset U \subset D(f(p)).$$

In a similar fashion, we can find two other real numbers $\mu, \nu \in (0,1]$ such that

$$f(p) \in f(\nu D(p)) \subset \mu D(f(p)) \subset f(\lambda D(p)) \subset D(f(p)).$$

Because $f(\nu D(p)) \subset \mu D(f(p))$, we can define the radial projection with center $f(p)$

$$\psi\colon\ f(\nu S(p)) \to \mu S(f(p))\ ,\ f(q) \mapsto \pi_{f(p)}(f(q))$$

for every $q \in \nu S(p)$. We also define the map

$$\phi\colon\ \mu S(f(p)) \to f(\nu S(p))\ ,\ q \mapsto f(\nu(\pi_p(f^{-1}(q)))$$

where π_p is the radial function with center p in $\lambda D(p)$ (notice that $f^{-1}(q) \neq p$, for every $q \in \mu S(f(p))$ and moreover, $f^{-1}(\mu D(f(p)) \subset \lambda D(p))$.

Since the spaces $f(\nu S(p))$ and $\mu S(f(p))$ are contained in $D(f(p))$ and this last space is $f(p)$-convex, we can define the homotopy

$$H_1\colon\ \mu S(f(p)) \times I \longrightarrow D(f(p))$$

$$H_1(q,t) = \begin{cases} (1-2t)\psi\phi(q) + 2tf(p)\ , & 0 \le t \le \frac{1}{2} \\ (2-2t)f(p) + (2t-1)\phi(q)\ , & \frac{1}{2} \le t \le 1 \end{cases}$$

for every $q \in \mu S(f(p))$. Strictly speaking, H_1 is a homotopy between ϕ composed with the inclusion map $f(\nu S(p) \subset D(f(p))$ and $\psi\phi$ composed with $\mu S(f(p)) \subset D(f(p))$. We now take the maps

$$f^{-1}\phi\colon\ \mu S(f(p)) \to \nu D(p) \text{ and } f^{-1}\colon\ \mu S(f(p)) \to \lambda D(p).$$

Because $D(p)$ is p-convex, we can construct the homotopy

$$H_2(q,t) = \begin{cases} (1-2t)f^{-1}\phi(q) + 2tp\ , & 0 \le t \le \frac{1}{2} \\ (2-2t)p + (2t-1)f^{-1}(q)\ , & \frac{1}{2} \le t \le 1\,. \end{cases}$$

which, when composed with the homeomorphism f, gives rise to a homotopy

$$fH_2\colon\ \mu S(f(p)) \times I \to D(f(p));$$

finally, we consider the homotopy

$$F\colon\ \mu S(f(p) \times I \to D(f(p))$$

defined by the formula

$$F(q,t) = \begin{cases} H_1(q,2t)\ , & 0 \le t \le \frac{1}{2} \\ fH_2(q,2t-1)\ , & \frac{1}{2} \le t \le 1\,. \end{cases}$$

The map F is a homotopy between $\psi\phi$ and the identity map of $\mu S(f(p))$. Similarly, we prove that $\phi\psi$ is homotopic to the corresponding identity map. Thus, $\mu S(f(p))$ and $f(\nu S(p))$ are of the same homotopy type. On the other hand, $\mu S(f(p))$ and $f(\nu S(p))$ are homeomorphic, respectively, to $S(f(p))$ and $S(p)$; hence, $S(f(p))$ and $S(p)$ are of the same homotopy type. ∎

The geometric realization $|K|$ of a simplicial complex K is called *polyhedron*.[2] The next theorem gives a better understanding of the topology of $|K| = (X,\Phi)$.

(II.2.12) Theorem. *A set $F \subset |K|$ is closed in $|K|$ if and only if, for every $\sigma \in \Phi$, the subset $F \cap |\overline{\sigma}|$ is closed in $|\overline{\sigma}|$.*

Proof. Because $|\overline{\sigma}|$ is a compact subset of a Hausdorff space $|K|$, $|\overline{\sigma}|$ is closed in $|K|$ (see Theorem (I.1.25)); thus, if F is closed in $|K|$, $F \cap |\overline{\sigma}|$ is closed in $|K|$ and therefore, F is closed in $|\overline{\sigma}|$.

Conversely, if $F \cap |\overline{\sigma}|$ is closed in $|\overline{\sigma}|$ for every $|\overline{\sigma}| \subset |K|$, then $F = \bigcup_{|\overline{\sigma}|}(F \cap |\overline{\sigma}|)$ is closed in $|K|$ as a finite union of closed sets. ∎

A topological space X is said to be *triangulable* if there exists a polyhedron K, which is homeomorphic to X; the simplicial complex K is a *triangulation* of X. A triangulable space can have more than one triangulation. For example, it is easy to understand that S^1 has a triangulation given by a simplicial complex whose geometric realization is homeomorphic to the boundary of an equilateral triangle; but it can also be triangulated by a complex whose geometric realization is a regular polygon with vertices in S^1 (the homeomorphisms are given by a projection from the center of S^1). More generally, a disk D^n and its boundary S^{n-1} are examples of triangulable spaces; these spaces also have several possible triangulations. Next, we describe the *standard triangulation* of S^n.

Let Σ^n be the set of all points $(x_1,x_2,\dots,x_{n+1}) \in \mathbb{R}^{n+1}$ such that $\sum_i |x_i| = 1$. Let X be the set of all vertices of Σ_n, that is to say, of the points $a_i = (0,\dots,\overset{i}{1},\dots,0)$ and $a'_i = (0,\dots,\overset{i}{-1},\dots,0)$ in $\mathbb{R}^{n+1}$, $i = 1,\dots,n+1$. Now, let Φ be the set of all nonempty subsets of X of the type $\{x_{i_0},\dots,x_{i_r}\}$ with $1 \leq i_0 < i_1 < \dots < i_r \leq n+1$ and x_{i_s} equal to either a_{i_s} or a'_{i_s}. Since any set of vertices of this type is linearly independent, $K^n = (X,\Phi)$ is a simplicial complex. Its geometric realization is homeomorphic to Σ^n; on the other hand, Σ^n and S^n are homeomorphic by a radial projection from the center and therefore, K^n is a triangulation of S^n. The simplicial complex K^n is the so-called *standard triangulation* of S^n.

(II.2.13) Remark. As we have already notice, in this book we work exclusively with finite simplicial complexes. However, it is possible to give a more extended definition of simplicial complexes, which includes the infinite case. With this in mind, we define a simplicial complex as a pair $K = (X,\Phi)$ in which X is a set

[2] In some textbooks, *polyhedra* are the geometric realizations of two-dimensional complexes; for the more general case, they use the word *polytopes*.

(not necessarily finite) and Φ is a set of nonempty, *finite* subsets of X satisfying the following properties:

1. $(\forall x \in X)$, $\{x\} \in \Phi$,
2. $(\forall \sigma \in \Phi)(\forall \sigma' \subset \sigma$, $\sigma' \neq \emptyset)$, $\sigma' \in \Phi$.

The price we must pay is a strengthening of the topology of K. We keep the metric

$$d\colon\ |K| \times |K| \longrightarrow \mathbb{R}_{\geq 0}$$
$$(\forall p,q \in |K|)\ d(p,q) = \left\{ \sum_{x \in X} (p(x) - q(x))^2 \right\}^{\frac{1}{2}}$$

which defines a topology on K. While the necessary condition of Theorem (II.2.12) is still valid, the sufficient condition does not hold because, to prove it, we need the assumption that X is finite. However, it is precisely the topology of Theorem (II.2.12) that we impose on $|K|$; in other words, we must exchange the metric topology of K with a finer topology. We say that

$F \subset |K|$ is *closed* $\iff$ $(\forall \sigma \in \Phi)$ $F \cap |\overline{\sigma}|$ is closed in σ.

This topology is normally called "weak topology"; this is somehow a strange name, considering the fact that the weak topology for K is finer (that is to say, has more open sets) than the metric topology.

II.2.2 Simplicial Complexes and Immersions

We have proved, aided by the geometric realization functor, that every abstract finite simplicial complex K can be immersed in an Euclidean space and hence can be viewed as an Euclidean simplicial complex. The dimension of the Euclidean space in question is equal to the number of vertices, say m, of the complex. At this point, we ask ourselves whether it is possible to immerse K in an Euclidean space of dimension lower than m. The next theorem answers that question. Before stating the theorem, we define Euclidean simplicial complexes in a different (but equivalent) fashion. Let $\mathfrak{K} \subset \mathbb{R}^n$ be a union of *finitely many* Euclidean simplexes of $\mathbb{R}^n$ such that

1. If $\sigma \subset \mathfrak{K}$, every face of σ is in $\mathfrak{K}$.
2. The intersection of any two Euclidean simplexes of $\mathfrak{K}$ is a face of both.

It is not difficult to prove that a set of simplexes of $\mathbb{R}^n$ verifying the previous conditions is an Euclidean simplicial complex as defined in Sect. II.1. We also notice that if $F \subset \mathfrak{K}$ is closed, the intersection $F \cap \sigma$ is closed in σ for every Euclidean simplex σ of $\mathfrak{K}$; conversely, if F is a subset of $\mathfrak{K}$ such that, for every Euclidean simplex σ of $\mathfrak{K}$, $F \cap \sigma$ is closed in σ, then F is closed in $\mathfrak{K}$ because F is the finite union of the closed sets $F \cap \sigma$. Clearly, an Euclidean complex $\mathfrak{K}$ of $\mathbb{R}^n$ is compact and closed in $\mathbb{R}^n$. We now state the immersion theorem for simplicial complexes.

(II.2.14) Theorem. *Every n-dimensional polyhedron $|K|$ is homeomorphic to an Euclidean simplicial complex.*

Proof. Let $\mathfrak{N}$ be the set of all points $P^i = (i, i^2, \ldots, i^{2n+1}) \in \mathbb{R}^{2n+1}$, for every $i \geq 0$. We claim that the set $\mathfrak{N}$ has the following property: every $2n+2$ points $P^{i_0}, \ldots, P^{i_{2n+1}}$ are linearly independent. In fact, a linear combination

$$\sum_{j=1}^{2n+1} \alpha_j (P^{i_j} - P^{i_0}) = 0$$

gives rise to the equations

$$\sum_{j=1}^{2n+1} \alpha_j = 0,$$
$$\sum_{j=1}^{2n+1} \alpha_j i_j^1 = 0,$$
$$\cdots$$
$$\sum_{j=1}^{2n+1} \alpha_j i_j^{2n+1} = 0;$$

because the determinant of the system of linear homogeneous equations defined by the $2n+2$ equations written above is equal to $\prod_{k>j}(i_k - i_j) \neq 0$, the only solution for the system is the trivial one, $\alpha_1 = \alpha_2 = \ldots = \alpha_{2n+1} = 0$.

Assume that $K = (X, \Phi)$ with $X = \{a_0, \ldots, a_s\}$ and $s \leq n$. To each vertex a_i, we associate the point $P^i = (i^1, i^2, \ldots, i^{2n+1}) \in \mathbb{R}^{2n+1}$, and to each simplex $\{a_{j_0}, a_{j_1}, \ldots, a_{j_p}\} \in \Phi$ we associate the Euclidean simplex $\{P^{j_0}, P^{j_1}, \ldots, P^{j_p}\}$ (observe that the points P^{j_i} with $j = 0, \ldots, p$ are linearly independent because $p \leq n < 2n+1$). Let $\mathfrak{K}$ be the set of vertices and Euclidean simplexes obtained in this way.

We begin by observing that $\mathfrak{K}$ clearly satisfies condition 1 of the definition of Euclidean simplicial complexes. Let us prove that condition 2 is also valid. Let σ_p and σ_q be two Euclidean simplexes of $\mathfrak{K}$ with r common vertices; altogether σ_p and σ_q have $p+q-r+2$ vertices. Because $p+q-r+2 \leq 2n+2$, these vertices form an Euclidean simplex of $\mathbb{R}^{2n+1}$ having σ_p and σ_q as faces; hence, $\sigma_p \cap \sigma_q$ is either empty (if $r = 0$) or a common face of σ_p and σ_q.

Therefore, $\mathfrak{K}$ is an Euclidean simplicial complex homeomorphic to $|K|$. ∎

The reader could ask whether Theorem (II.2.14) is the best possible result or else, whether it is possible to realize all n-dimensional simplicial complexes in Euclidean spaces of dimension less than $2n+1$. Clearly, a complex of dimension n must be immersed in a space of dimension at least n. We shall now give two examples of one-dimensional simplicial complexes (that is to say, graphs) that cannot be immersed in $\mathbb{R}^2$.

Let $K_{3,3}$ be the *complete bipartite graph* over two sets of 3 vertices, also known as *utility graph*: $K_{3,3} = (X, \Phi)$ with $X = \{1,2,3,a,b,c\}$ and

$$\Phi = \{\{1\},\{2\},\{3\},\{a\},\{b\},\{c\},\{1,a\},\{1,b\},\{1,c\},\{2,a\},\{2,b\},\{2,c\},\{3,a\},\{3,b\},\{3,c\}\}.$$

Figure II.7 shows its graphic representation (which however is *not* a geometric real-

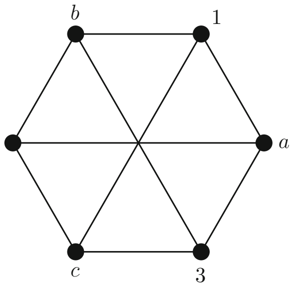

Fig. II.7

ization of $K_{3,3}$ because its distinct 1-simplexes have empty intersections). Another way to represent $K(3,3)$ is given in Fig. II.8. To ask whether or not $K(3,3)$ can be

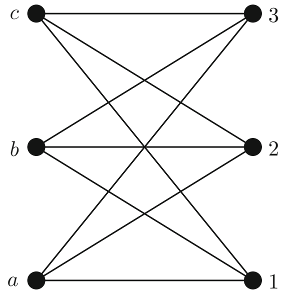

Fig. II.8

represented as a *planar* graph is a classical query; the answer would be affirmative if one could determine an immersion of $K(3,3)$ in $\mathbb{R}^2$ (but planarity is a weaker property: It is enough to show that $|K_{3,3}|$ is homeomorphic to a subspace of $\mathbb{R}^2$). Another example of a simplicial complex with an analogous property is given by the *complete graph* over 5 vertices K_5: $X = \{1,2,3,4,5\}$ and Φ is the set of all nonempty subsets of X with at most 2 elements. Figure II.9 is a standard graphic representation of this graph. It is not difficult to prove that both $K_{3,3}$ and K_5 cannot be immersed in $\mathbb{R}^2$.

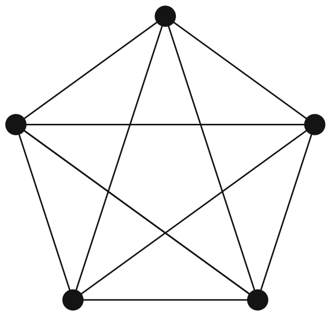

Fig. II.9

II.2.3 The Homology Functor

We now define the *homology functor*

$$H_*(-;\mathbb{Z})\colon\ \mathbf{Csim} \to \mathbf{Ab}^{\mathbb{Z}},$$

another important functor with domain **Csim**.

Let $K = (X, \Phi)$ be an arbitrary simplicial complex. We begin our work by giving an *orientation* to the simplexes of K. Let $\sigma_n = \{x_0, x_1, \ldots, x_n\}$ be an n-simplex of K; the elements of σ_n can be ordered in $(n+1)!$ different ways. We say that two orderings of the elements of σ_n are *equivalent* whenever they differ by an even permutation; an *orientation* of σ_n is an equivalence class of orderings of the vertices of σ_n, provided that $n > 0$. An n-simplex $\sigma_n = \{x_0, x_1, \ldots, x_n\}$ has two orientations. A 0-simplex has clearly only one ordering; its orientation is given by ± 1.

If $\sigma_n = \{x_0, x_1, \ldots, x_n\}$ is oriented, the simplex $\{x_1, x_0, \ldots, x_n\}$ for example, is denoted with $-\sigma$. If $n \geq 1$, a given orientation of $\sigma_n = \{x_0, x_1, \ldots, x_n\}$ automatically defines an orientation in all of its $(n-1)$-faces: For example, if $\sigma_2 = \{x_0, x_1, x_2\}$ is oriented by the ordering $x_0 < x_1 < x_2$, its oriented 1-faces are

$$\{x_1, x_2\}\ ,\ \{x_2, x_0\} = -\{x_0, x_2\} \text{ and } \{x_0, x_1\}.$$

More generally, if $\sigma_n = \{x_0, x_1, \ldots, x_n\}$ is oriented by the natural ordering of the indices of its vertices, its $(n-1)$-face

$$\sigma_{n-1,i} = \{x_0, x_1, \ldots, \widehat{x_i}, \ldots, x_n\} = \{x_0, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\}$$

(opposite to the vertex x_i with $i = 0, \ldots, n$) has an orientation given by $(-1)^i \sigma_{n-1,i}$; we say that $\sigma_{n-1,i}$ is *oriented coherently* to σ_n if i is even, and is oriented coherently to $-\sigma_n$ if i is odd. We observe explicitly that the symbol $\widehat{\ }$ over the vertex x_i means that such vertex has been eliminated.

We are now ready to order a simplicial complex $K = (X, \Phi)$. We recall that the technique used to give an orientation to a simplex was first to order its vertices in all possible ways, and then choose an ordering class (there are two possible classes: the class in which the orderings differ by an even permutation, and that in which the orderings differ by an odd permutation). Now let us move to K. Begin by taking a partial ordering of the set X in such a way that the set of vertices of each simplex $\sigma \in \Phi$ is totally ordered; in this way, we obtain an ordering class – that is to say, an orientation – for each simplex. A simplicial complex whose simplexes are all oriented is said to be *oriented*.

Let $K = (X, \Phi)$ be an oriented simplicial complex. For every $n \in \mathbb{Z}$, with $n \geq 0$, let $C_n(K)$ be the free Abelian group defined by all linear combinations with coefficients in $\mathbb{Z}$ of the oriented n-simplexes of K; in other words, if $\{\sigma_n^i\}$ is the finite set of all oriented n-simplexes of K, then $C_n(K)$ is the set of all formal sums $\sum_i m_i \sigma_n^i$, $m_i \in \mathbb{Z}$ (called *n-chains*), together with the addition law

$$\sum_i p_i \sigma_n^i + \sum_i q_i \sigma_n^i := \sum_i (p_i + q_i) \sigma_n^i.$$

If $n < 0$, we set $C_n(K) = 0$. Now, for every $n \in \mathbb{Z}$, we define a homomorphism $\partial_n = \partial_n^K \colon C_n(K) \longrightarrow C_{n-1}(K)$ as follows: if $n \leq 0$, ∂_n is the constant homomorphism 0; if $n \geq 1$, we first define ∂_n over an oriented n-simplex $\{x_0, x_1, \dots, x_n\}$ (viewed as an n-chain) as

$$\partial_n(\{x_0, x_1, \dots, x_n\}) = \sum_{i=0}^{n} (-1)^i \{x_0, \dots, \widehat{x_i}, \dots, x_n\} ;$$

finally, we extend this definition by linearity over an arbitrary n-chain of oriented n-simplexes. The homomorphisms of degree -1, that we have just defined, are called *boundary homomorphisms*.

(II.2.15) Lemma. *For every $n \in \mathbb{Z}$, the composition $\partial_{n-1}\partial_n = 0$.*

Proof. The result is obvious if $n = 1$. Let $\{x_0, x_1, \dots, x_n\}$ be an arbitrary oriented n-simplex with $n \geq 2$. Then

$$\begin{aligned}
\partial_{n-1}\partial_n(\{x_0, x_1, \dots, x_n\}) &= \partial_{n-1} \sum_{i=0}^{n} (-1)^i \{x_0, \dots, \widehat{x_i}, \dots, x_n\} \\
&= \sum_{j<i} (-1)^i (-1)^j \{x_0, \dots, \widehat{x_j}, \dots, \widehat{x_i}, \dots, x_n\} \\
&+ \sum_{j>i} (-1)^i (-1)^{j-1} \{x_0, \dots, \widehat{x_i}, \dots, \widehat{x_j}, \dots, x_n\}.
\end{aligned}$$

This summation is 0 because its addendum $\{x_0, \dots, \widehat{x_j}, \dots, \widehat{x_i}, \dots, x_n\}$ appears twice, once with the sign $(-1)^i(-1)^j$ and once with the sign $(-1)^i(-1)^{j-1}$. ■

This important property of the boundary homomorphisms implies that, for every $n \in \mathbb{Z}$, the image of ∂_{n+1} is contained in the kernel of ∂_n; using the notation $Z_n(K) = \ker \partial_n$ and $B_n(K) = \operatorname{im} \partial_{n+1}$, we conclude that $B_n(K) \subset Z_n(K)$ for every $n \in \mathbb{Z}$. Thus, to each integer $n \geq 0$, we can associate the quotient group

$$H_n(K; \mathbb{Z}) = Z_n(K)/B_n(K);$$

to each $n < 0$, we associate $H_n(K; \mathbb{Z}) = 0$.

(II.2.16) Definition. An n-chain $c_n \in C_n(K)$ is an *n-cycle* (or simply cycle) if $\partial_n(c_n) = 0$; thus $Z_n(K)$ is the set of all n-cycles. An n-chain c_n, for which we can find an $(n+1)$-chain c_{n+1} such that $c_n = \partial_{n+1}(c_{n+1})$, is an *n-boundary*; thus, $B_n(K)$ is the set of all n-boundaries. Two n-chains c_n and c'_n are said to be *homologous* if $c_n - c'_n \in B_n(K)$.

What we have just described is a method to associate a graded Abelian group

$$H_*(K;\mathbb{Z}) = \{H_n(K;\mathbb{Z}) \,|\, n \in \mathbb{Z}\}$$

to any oriented simplicial complex $K \in$ **Csim**. To define a functor on **Csim** we must see what happens to the morphisms; we proceed as follows. Let $f\colon\ K = (X,\Phi) \to L = (Y,\Psi)$ be a simplicial function (K and L have a fixed orientation). We first define

$$C_n(f)\colon\ C_n(K) \longrightarrow C_n(L)$$

on the simplexes by

$$C_n(f)(\{x_0,\dots,x_n\}) = \begin{cases} \{f(x_0),\dots,f(x_n)\}, & (\forall i \neq j)\ f(x_i) \neq f(x_j) \\ 0, & \text{otherwise} \end{cases}$$

and then extend $C_n(f)$ linearly over the whole Abelian group $C_n(K)$. It is easy to prove that $\partial^L_n C_n(f) = C_{n-1}(f)\partial^K_n$, for every $n \in \mathbb{Z}$ (one can verify this on a single n-simplex). We now define

$$H_n(f)\colon\ H_n(K;\mathbb{Z}) \longrightarrow H_n(L;\mathbb{Z})$$
$$z + B_n(K) \mapsto C_n(f)(z) + B_n(L)$$

for every $n \geq 0$. We begin by observing that $C_n(f)(z)$ is a cycle in $C_n(L)$: in fact, since z is a cycle,

$$\partial^L_n C_n(f)(z) = C_{n-1}(f)\partial^K_n(z) = 0\ .$$

On the other hand, we note that $H_n(f)$ is well defined: let us assume that $z - z' = \partial^K_{n+1}(w)$; then

$$C_n(f)(z-z') = C_n(f)\partial^K_{n+1}(w) = \partial^L_{n+1}C_{n+1}(f)(w)$$

and thus $C_n(f)(z-z') \in B_n(L)$; from this, we conclude that $H_n(f)((z-z') + B_n(K)) = 0$.

If $n < 0$, we set $H_n(f) = 0$; in this way, we obtain a homomorphism $H_n(f)$ between Abelian groups, for every $n \in \mathbb{Z}$. The reader is invited to prove that

$$H_n(1_K) = 1_{H_n(K)} \text{ e } H_n(gf) = H_n(g)H_n(f)$$

for every $n \in \mathbb{Z}$.

(II.2.17) Remark. The construction of the *homology groups* $H_n(K;\mathbb{Z})$ is independent from the orientation of K, up to isomorphism. In fact, suppose that O and O'

are two distinct orientations of K and denote by K^O and $K^{O'}$ the complex K together with the orientations O and O', respectively.

The simplexes of K^O are denoted by σ, and those of $K^{O'}$, by σ'. Now define $\phi_n : C_n(K^O) \to C_n(K^{O'})$ as the function taking a simplex σ_n into the simplex σ_n' if O and O' give the same orientation to σ_n, and taking σ_n into $-\sigma_n'$ if O and O' give opposite orientations to σ_n; next, extend ϕ_n by linearity over the whole group $C_n(K^O)$. It is easy to prove that ϕ_n is a group isomorphism. Moreover, for every $n \in \mathbb{Z}$, $\partial_n\phi_n = \phi_{n-1}\partial_n$. For a given n-simplex σ_n of K^O, we have two cases to consider:

Case 1: O and O' give the same orientation to σ_n; then

$$\partial_n\phi_n(\sigma_n) = \partial_n(\sigma_n') = \sum_{i=0}^{n}(-1)^i\sigma'_{n-1,i}$$

$$\phi_{n-1}\partial_n(\sigma_n) = \phi_{n-1}(\sum_{i=0}^{n}(-1)^i\sigma_{n-1,i}) = \sum_{i=0}^{n}(-1)^i\sigma'_{n-1,i};$$

Case 2: O and O' give different orientations to σ_n; then

$$\partial_n\phi_n(\sigma_n) = \partial_n(-\sigma_n') = \sum_{i=0}^{n}(-1)^{i+1}\sigma'_{n-1,i}$$

$$\phi_{n-1}\partial_n(\sigma_n) = \phi_{n-1}(\sum_{i=0}^{n}(-1)^{i+1}\sigma_{n-1,i}) = \sum_{i=0}^{n}(-1)^{i+1}\sigma'_{n-1,i}.$$

Similar to what we did to define the homomorphism $H_n(f)$, we can prove that ϕ_n induces a homomorphism

$$H_n(\phi_n)\colon\ H_n(K^O;\mathbb{Z}) \longrightarrow H_n(K^{O'};\mathbb{Z})$$

which is actually an isomorphism.

Therefore, up to isomorphism, the orientation given to a simplicial complex has no influence on the definition of the group $H_n(K;\mathbb{Z})$; thus, we forget the orientation (however, we note that in certain questions it cannot be ignored). With this, we define the covariant functor

$$H_*(-;\mathbb{Z})\colon\ \mathbf{Csim} \longrightarrow \mathbf{Ab}^{\mathbb{Z}}$$

by setting

$$H_*(K;\mathbb{Z}) = \{H_n(K;\mathbb{Z}) \mid n \in \mathbb{Z}\} \text{ and } H_*(f) = \{H_n(f) \mid n \in \mathbb{Z}\}$$

on objects and morphisms, respectively. The graduate Abelian group $H_*(K;\mathbb{Z})$ is the *(simplicial) homology* of K with coefficients in $\mathbb{Z}$.

We are going to compute the homology groups of the simplicial complex T^2 depicted in Fig. II.10 and whose geometric realization is the two-dimensional torus. We begin by orienting T^2 so that we go clockwise around the boundary of each 2-simplex. To simplify the notation, let us write $C_i(T^2)$ as C_i (the same for the

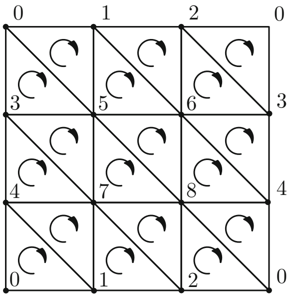

Fig. II.10 A triangulation of the torus with oriented simplexes

groups of boundaries and cycles). We notice that $C_2 \cong \mathbb{Z}^{18}$, $C_1 \cong \mathbb{Z}^{27}$, $C_0 \cong \mathbb{Z}^9$. We represent the boundary homomorphisms in the next diagram

$$0 \longrightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0\,.$$

Clearly, each vertex (and hence, each 0-chain) is a cycle; hence, $Z_0 = C_0$. The elements $\{0\}$, $\{1\}-\{0\}$, $\dots \{8\}-\{0\}$ form a basis of Z_0. Any two vertices can be connected by a sequence of 1-simplexes and so the 0-cycles $\{1\}-\{0\}$, ..., $\{8\}-\{0\}$ are 0-boundaries. Since the boundary of a generic 1-simplex $\{i,j\}$ can be written as

$$\partial_1(\{i,j\}) = \{j\}-\{i\} = \{j\}-\{0\}-(\{i\}-\{0\}),$$

we have that $B_0 \subset Z_0$ is generated by $\{1\}-\{0\},\dots,\{8\}-\{0\}$ and thus,

$$H_0(T^2;\mathbb{Z}) \cong \mathbb{Z}\,.$$

The homology class of any vertex is a generator of this group.

Next, we compute $H_1(T^2;\mathbb{Z})$. The two 1-chains

$$z_1^1 = \{0,3\}+\{3,4\}+\{4,0\} \text{ and } z_1^2 = \{0,1\}+\{1,2\}+\{2,0\}$$

are cycles and generate (in Z_1) a free Abelian group of rank 2 which we denote by $S \cong \mathbb{Z}\oplus\mathbb{Z}$. Let $z \in Z_1$ be a 1-cycle $z = \sum_i k_i\sigma_1^i$, in which σ_1^i are the 1-simplexes and $k_i \in \mathbb{Z}$. By adding suitable multiples of 2-simplexes, it is possible to find a 1-boundary b such that the 1-cycle $z-b$ does not contain the terms, which correspond to the diagonal 1-simplexes $\{0,5\}$, $\{1,6\}$, ..., $\{7,2\}$, $\{8,0\}$. Similarly, adding suitable pairs of adjacent 2-simplexes (those forming squares with a common diagonal) it is possible to find a 1-boundary b' such that the cycle $z-b-b'$ contains only the terms corresponding to the 1-simplexes $\{0,3\}$, $\{3,4\}$, $\{4,0\}$,$\{0,1\}$, $\{1,2\}$, and $\{2,0\}$ (we leave the details to the reader, as an exercise). Because $z-b-b'$ is a 1-cycle, it follows that $z-b-b' \in S$. This argument shows that $B_1+S = Z_1$. Let us now suppose that $B_1\cap S \neq 0$; then there exists a linear combination of 2-simplexes $\sum_j h_j\sigma_2^j$ such that $\sum_j h_j\partial\sigma_2^j \in S$. If two 2-simplexes σ_2^i and σ_2^j have a common

1-simplex "internal" to the square of Fig. II.10, then they must have equal coefficients $h_i = h_j$. This implies that there exists $h \in \mathbb{Z}$ such that $h_j = h$ for each j, that is to say, $\sum_j h_j \sigma_2^j = hz_2$ where z_2 is the 2-chain $\sum_j \sigma_2^j$. It is easy to see that $\partial z_2 = \sum_j \partial \sigma_2^j = 0$ and so $B_1 \cap S = 0$, implying that $H_1(T^2;\mathbb{Z}) \cong S \cong \mathbb{Z}^2$, with free generators z_1^1 and z_1^2.

Finally, similar arguments show that any 2-cycle of C_2 is a multiple of the 2-chain z_2 defined above (given by the sum of all oriented 2-simplexes of T^2) and therefore, $H_2(T^2;\mathbb{Z}) \cong \mathbb{Z}$.

Exercises

1. Let $\mathscr{U} = \{U_x | x \in X\}$ be a *finite* open covering of a topological space B, and take the set

$$\Phi = \{\sigma \subset X | \bigcap_{x\in\sigma} U_x \neq \emptyset\}.$$

Prove that $N(\mathscr{U}) = (X, \Phi)$ is a simplicial complex. This is the so-called *nerve* of $\mathscr{U}$.

2. Let $K = (X, \Phi)$ be a simplicial complex. For a given $x \in X$, let $St(x)$ be the complement in $|K|$ of the union of all $|\overline{\sigma}|$ such that $x \notin \sigma$, $\sigma \in \Phi$. $St(x)$ is called *star* of x in $|K|$. Prove that $\mathscr{S} = \{St(x) \,|\, x \in X\}$ is an open covering of $|K|$, and $N(\mathscr{S}) = K$.

3. Let X be a compact metric space and let ε be a positive real number. Take the set Φ of all *finite* subsets of X with diameter less than ε. Prove that $K = (X, \Phi)$ is a simplicial complex (infinite).

4. Exhibit a triangulation of the following spaces:

a) *Cylinder C* – recall that the cylinder C is obtained from a rectangle by identification of two opposite sides;
b) *Möbius band M* obtained from a rectangle by identification of the "inverse" points of two opposite sides; more precisely, let S be the rectangle with vertices $(0,0)$, $(0,1)$, $(2,0)$, and $(2,1)$ of $\mathbb{R}^2$; then

$$M = S/\{(0,t) \equiv (2, 1-t)\}\ ,\ 0 \leq t \leq 1;$$

c) *Klein bottle K* obtained by identifying the "inverse" points of the boundary of the cylinder C;
d) *real projective plane* $\mathbb{R}P^2$ obtained by the identification of the antipodal points of the boundary $\partial D^2 \cong S^1$ of the unit disk $D^2 \subset \mathbb{R}^2$;
e) G_2 obtained by attaching two handles to the sphere S^2; prove that G_2 is homeomorphic to the space obtained from an octagon with the suitable identifications of the edges of its border $a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1}$.

II.3 Introduction to Homological Algebra

In the previous section, we have seen that we can associate a graded Abelian group $C(K) = \{C_n(K)\}$ with any simplicial complex K and a homomorphism $\partial_n\colon C_n(K) \to C_{n-1}(K)$, such that $\partial_{n-1}\partial_n = 0$, to each integer n; these homomorphisms define a graded Abelian group $H_*(K;\mathbb{Z}) = \{H_n(K;\mathbb{Z})\}$. All this can be viewed in the framework of a more general and more useful context.

A *chain complex* (C,∂) is a graded Abelian group $C = \{C_n\}$ together with an endomorphism $\partial = \{\partial_n\}$ of degree -1, called *boundary homomorphism*[3] $\partial = \{\partial_n\colon C_n \to C_{n-1}\}$, such that $\partial^2 = 0$; this means that, for every $n \in \mathbb{Z}$, $\partial_n\partial_{n+1} = 0$. Hence

$$B_n = \operatorname{im}\partial_{n+1} \subset Z_n = \ker\partial_n$$

and so we can define the graded Abelian group

$$H_*(C) = \{H_n(C) = Z_n/B_n \mid n \in \mathbb{Z}\};$$

this is the *homology* of C.

A *chain homomorphism* between two chain complexes (C,∂) and (C',∂') is a graded group homomorphism $f = \{f_n\colon C_n \to C'_n\}$ of degree 0 commuting with the boundary homomorphism, that is to say, for every $n \in \mathbb{Z}$, $f_{n-1}\partial_n = \partial'_n f_n$.

Chain complexes and chain homomorphisms form a category $\mathfrak{C}$, the *category of chain complexes*.

It is costumary to visualize chain complexes as diagrams

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots$$

and their morphisms as commutative diagrams

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\
\cdots & \longrightarrow & C'_{n+1} & \xrightarrow{\partial'_{n+1}} & C'_n & \xrightarrow{\partial'_n} & C'_{n-1} & \longrightarrow & \cdots
\end{array}$$

The previous definitions are clearly inspired by what we did to define the homology groups of a simplicial complex; indeed, we emphasize the fact that, for every simplicial complex X, the graded Abelian group $\{C_n(K)|n \in \mathbb{Z}\}$ together with its boundary homomorphism $\partial^K = \{\partial_n^K|n \in \mathbb{Z}\}$ is a chain complex $(C(K),\partial^K)$. The chain complex $C(K)$ is said to be *positive* because its terms of negative index are 0. In particular, for every simplicial function $f\colon K \to M$, the homomorphism

$$C(f)\colon C(K) \to C(M)$$

is a chain homomorphism.

[3] In some textbooks, it is called *differential operator*.

An infinite sequence of Abelian groups

$$\cdots \longrightarrow G_{n+1} \xrightarrow{f_{n+1}} G_n \xrightarrow{f_n} G_{n-1} \longrightarrow \cdots$$

is said to be *exact* if and only if, for every $n \in \mathbb{Z}$, $\operatorname{im} f_{n+1} = \ker f_n$.

The exact sequences with only three consecutive nontrivial groups

$$\cdots \longrightarrow 0 \longrightarrow G_{n+1} \xrightarrow{f_{n+1}} G_n \xrightarrow{f_n} G_{n-1} \longrightarrow 0 \longrightarrow \cdots$$

are particularly important; in that case, f_{n+1} is injective and f_n is surjective. These sequences are called *short exact sequences*. The previous short exact sequence is also written up in the form

$$G_{n+1} \overset{f_{n+1}}{\rightarrowtail} G_n \overset{f_n}{\twoheadrightarrow} G_{n-1}.$$

The concept of short exact sequence of groups can be easily exported to the category $\mathfrak{C}$ of chain complexes: a sequence of chain complexes

$$(C,\partial) \overset{f}{\rightarrowtail} (C',\partial') \overset{g}{\twoheadrightarrow} (C'',\partial'')$$

is *exact* if every horizontal line of its representative diagram

$$\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
\downarrow{\scriptstyle \partial_{n+2}} & & \downarrow{\scriptstyle \partial'_{n+2}} & & \downarrow{\scriptstyle \partial''_{n+2}} \\
C_{n+1} & \overset{f_{n+1}}{\rightarrowtail} & C'_{n+1} & \overset{g_{n+1}}{\twoheadrightarrow} & C''_{n+1} \\
\downarrow{\scriptstyle \partial_{n+1}} & & \downarrow{\scriptstyle \partial'_{n+1}} & & \downarrow{\scriptstyle \partial''_{n+1}} \\
C_n & \overset{f_n}{\rightarrowtail} & C'_n & \overset{g_n}{\twoheadrightarrow} & C''_n \\
\downarrow{\scriptstyle \partial_n} & & \downarrow{\scriptstyle \partial'_n} & & \downarrow{\scriptstyle \partial''_n} \\
C_{n-1} & \overset{f_{n-1}}{\rightarrowtail} & C'_{n-1} & \overset{g_{n-1}}{\twoheadrightarrow} & C''_{n-1} \\
\downarrow{\scriptstyle \partial_{n-1}} & & \downarrow{\scriptstyle \partial'_{n-1}} & & \downarrow{\scriptstyle \partial''_{n-1}} \\
\vdots & & \vdots & & \vdots
\end{array}$$

is exact and each square is commutative.

The next result is very important; it is the so-called **Long Exact Sequence Theorem**.

(II.3.1) Theorem. *Let*

$$(C,\partial) \overset{f}{\rightarrowtail} (C',\partial') \overset{g}{\twoheadrightarrow} (C'',\partial'')$$

be a short exact sequence of chain complexes. For every $n \in \mathbb{Z}$, there exists a homomorphism

$$\lambda_n\colon\ H_n(C'') \to H_{n-1}(C)$$

(called connecting homomorphism) making exact the following sequence of homology groups

$$\cdots \longrightarrow H_n(C) \xrightarrow{H_n(f)} H_n(C') \xrightarrow{H_n(g)} H_n(C'') \xrightarrow{\lambda_n} H_{n-1}(C) \longrightarrow \cdots.$$

Proof. The proof of this theorem is not difficult. However, it is very long; we shall divide it into several steps, leaving some of the proofs to the reader, as exercises.

1. *Definition of λ_n.* Take the following portion of the short exact sequence of chain complexes:

$$\begin{array}{ccccc} C_n & \overset{f_n}{\rightarrowtail} & C'_n & \overset{g_n}{\twoheadrightarrow} & C''_n \\ {\scriptstyle \partial_n}\downarrow & & {\scriptstyle \partial'_n}\downarrow & & {\scriptstyle \partial''_n}\downarrow \\ C_{n-1} & \underset{f_{n-1}}{\rightarrowtail} & C'_{n-1} & \underset{g_{n-1}}{\twoheadrightarrow} & C''_{n-1} \end{array}$$

Let z be a cycle of C''_n; since g_n is surjective, there exists a chain $\tilde{z} \in C'_n$ such that $g_n(\tilde{z}) = z$. Because the diagram is commutative,

$$g_{n-1}\partial'_n(\tilde{z}) = \partial''_n g_n(\tilde{z}) = \partial''_n(z) = 0$$

and thus, $\partial'_n(\tilde{z}) \in \ker g_{n-1} = \operatorname{im} f_{n-1}$; hence, there exists a unique chain $c \in C_{n-1}$ such that

$$f_{n-1}(c) = \partial'_n(\tilde{z}).$$

Actually, c is a cycle because

$$f_{n-2}\partial_{n-1}(c) = \partial'_{n-1}f_{n-1}(c) = \partial'_{n-1}\partial'_n(\tilde{z}) = 0$$

and f_{n-2} is a monomorphism. It follows that we can define

$$\lambda_n\colon\ H_n(C'') \to H_{n-1}(C)$$

by setting $\lambda_n[z] := [c]$.

2. λ_n *is well defined.* We must verify that λ_n is independent from both the choice of the cycle z representing the homology class and the chain $\tilde{z}$ mapped into z. Let $z' \in C''$ be a cycle such that $[z] = [z']$, and let $\tilde{z}' \in C'_n$ be such that $g_n(\tilde{z}') = z'$; moreover, take a cycle c' in C'_{n-1} satisfying the property $f_{n-1}(c') = \partial'_n(\tilde{z}')$. The definition of homology classes implies that there exists a chain $b \in C''_{n+1}$ such that $\partial''_{n+1}(b) = z - z'$. Since g_{n+1} is an epimorphism, we can find a $\tilde{b} \in C'_{n+1}$ such that $g_{n+1}(\tilde{b}) = b$. Hence,

$$g_n(\tilde{z} - \tilde{z}' - \partial'_{n+1}(\tilde{b})) = z - z' - \partial''_{n+1}(b) = 0$$

and thus, there exists $a \in C_n$ such that

$$f_n(a) = \tilde{z} - \tilde{z}' - \partial'_{n+1}(\tilde{b}).$$

At this point, we have that

$$f_{n-1}(\partial_n(a)) = \partial'_n(\tilde{z} - \tilde{z}' - \partial'_{n+1}(\tilde{b})) = f_{n-1}(c - c')$$

and because f_{n-1} is injective, we conclude that $c - c' = \partial_n(a)$. Therefore, c and c' represent the same homology class in $H_n(C)$.

(II.3.2) Remark. The previous items 1. and 2. are typical examples of the so-called *"diagram chasing"* technique. We suggest the reader to draw the diagrams indicating the maps without their indices which, although necessary for precision, are sometimes difficult to read; all this will help in following up the arguments.

3. *The sequence is exact.* To prove the exactness of the sequence of homology groups, we must show the following:

   a. $\operatorname{im} H_n(f) = \ker H_n(g)$;
   b. $\operatorname{im} H_n(g) = \ker \lambda_n$;
   c. $\operatorname{im} \lambda_n = \ker H_{n-1}(f)$.

   We shall only prove (b), leaving the proof of the other cases to the reader. We pick a class $[z] \in H_n(C'')$ and compute $\lambda_n H_n(g)([z]) = \lambda_n[g_n(z)]$. Since we can take any (!) element of C'_n which is projected onto $g_n(z)$, we choose z itself; given that $\partial''_n(z) = 0$, we conclude that $\lambda_n[g_n(z)] = 0$, that is to say, $\operatorname{im} H_n(g) \subseteq \ker \lambda_n$. Conversely, let $[z]$ be a homology class of $H_n(C'')$ such that $\lambda_n[z] = 0$; the definition of λ_n implies that there exist $\tilde{z} \in C'_n$ and a cycle $c \in C_{n-1}$ such that

$$g_n(\tilde{z}) = z \text{ and } C_{n-1}(f)(c) = \partial'_n(\tilde{z}).$$

   Because $\lambda_n[z] = 0$, there exists $\tilde{c} \in C_n$ such that $c = \partial_n(\tilde{c})$. Notice that

$$\partial'_n(C_n(f)(\tilde{c}) - \tilde{z}) = 0$$

   and moreover, $H_n(g)(C_n(g)(\tilde{c}) - \tilde{z}) = z$; hence, $\ker \lambda_n \subseteq \operatorname{im} H_n(g)$.

■

The connecting homomorphisms λ_n are *natural* in the following sense:

(II.3.3) Theorem. *Let*

$$\begin{array}{ccccc}
(C,\partial) & \overset{f}{\rightarrowtail} & (C',\partial') & \overset{g}{\twoheadrightarrow} & (C'',\partial'') \\
\downarrow h & & \downarrow k & & \downarrow \ell \\
(\bar{C},\bar{d}) & \underset{\bar{f}}{\rightarrowtail} & (\bar{C'},\bar{d'}) & \underset{\bar{g}}{\twoheadrightarrow} & (\bar{C''},\bar{d''})
\end{array}$$

be a commutative diagram of chain complexes in which the horizontal lines are short exact sequences. Then, for every $n \in \mathbb{Z}$, the next diagram commutes.

$$\begin{array}{ccc}
H_n(C'') & \xrightarrow{\lambda_n} & H_{n-1}(C) \\
H_n(\ell)\downarrow & & \downarrow H_{n-1}(h) \\
H_n(\bar{C''}) & \xrightarrow[\bar{\lambda}_n]{} & H_{n-1}(\bar{C})
\end{array}$$

The proof of this theorem is easy and is left to the reader.

Let $f,g\colon (C,\partial) \to (C',\partial')$ be chain complex morphisms. We say that f and g are *chain homotopic* if there is a graded group morphism of degree $+\,1$, $s\colon C \to C'$ such that $f - g = d's + sd$; more precisely

$$(\forall n \in \mathbb{Z})\ f_n - g_n = \partial'_{n+1} s_n + s_{n-1}\partial_n\ .$$

The morphism $s\colon C \to C'$ is a *chain homotopy* between f and g (or from f to g). Notice that the chain homotopy relation just defined is an equivalence relation in the set

$$\mathfrak{C}((C,\partial),(C',\partial'))\ .$$

In particular, a morphism $f \in \mathfrak{C}((C,\partial),(C',\partial'))$ is *chain null-homotopic* if there exists a chain homotopy s such that $f = d's + sd$ (it follows that f and g are chain homotopic if and only if $f - g$ is chain null-homotopic).

(II.3.4) Proposition. *If $f,g \in \mathfrak{C}((C,\partial),(C',\partial'))$ are chain homotopic, then*

$$(\forall n)\ H_n(f) = H_n(g)\colon\ H_nC \to H_nC'.$$

Proof. For any cycle $z \in Z_nC$, we have that

$$H_nf[z] = [f_n(z)] = [g_n(z)] + [\partial'_{n+1}s_n(z)] + [s_{n-1}\partial_n(z)] = H_ng[z];$$

we now notice that $\partial_n z = 0$ and that $\partial'_{n+1}s_n(z)$ is a boundary and thus, homologous to zero. ■

Please notice that if f is chain null-homotopic, then $H_n(f) = 0$ for every n.

A chain complex (C, ∂) is *free* if all of its groups are free Abelian; it is *positive* if $C_n = 0$ for every $n < 0$. A positive chain complex (C, ∂) is *augmented* (to $\mathbb{Z}$) if there exists an epimorphism

$$\varepsilon\colon\ C_0 \to \mathbb{Z}$$

such that $\varepsilon\partial_1 = 0$. The homomorphism ε is the *augmentation (homomorphism).*

(II.3.5) Remark. The chain complex $C(K)$ associated with a simplicial complex K is free and positive. Moreover, the function

$$\varepsilon\colon\ C_0(K) \to \mathbb{Z},\ \ \Sigma_{i=1}^n a_i\{x_i\} \mapsto \Sigma_{i=1}^n a_i$$

is an augmentation.

A chain complex (C, ∂) is *acyclic* if, for every $n \in \mathbb{Z}$, $\ker\partial_n = \operatorname{im}\partial_{n+1}$, that is to say, if the sequence

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots$$

is exact. A positive chain complex (C, ∂) with augmentation is *acyclic* if the sequence

$$\cdots \longrightarrow C_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_1} C_0 \xrightarrow{\varepsilon} \mathbb{Z}$$

is exact.

Let (C, ∂) and (C', ∂') be two positive augmented chain complexes. A morphism $f \in \mathfrak{C}((C, \partial), (C', \partial'))$ is an *extension* of a homomorphism $\bar{f}\colon\ \mathbb{Z} \to \mathbb{Z}$ if the next diagram commutes.

$$\begin{array}{ccccccc} \cdots \longrightarrow & C_1 & \xrightarrow{\partial_1} & C_0 & \xrightarrow{\varepsilon} & \mathbb{Z} \\ & \downarrow f_1 & & \downarrow f_0 & & \downarrow \bar{f} \\ \cdots \longrightarrow & C'_1 & \xrightarrow{\partial'_1} & C'_0 & \xrightarrow{\varepsilon'} & \mathbb{Z} \end{array}$$

(II.3.6) Theorem. *Let (C, ∂) and (C', ∂') be positive augmented chain complexes; assume that (C, ∂) is free and (C', ∂') is acyclic. Then any homomorphism $\bar{f}\colon\ \mathbb{Z} \to \mathbb{Z}$ admits an extension $f\colon\ (C, \partial) \to (C', \partial')$, unique up to chain homotopy.*

Proof. Since the augmentation $\varepsilon'\colon\ C'_0 \to \mathbb{Z}$ is surjective, for every basis element x_0 of C_0, we choose an element of C'_0 which is taken onto $\bar{f}\varepsilon(x_0)$ by ε'; in this way, we obtain a homomorphism $f_0\colon\ C_0 \to C'_0$ such that $\bar{f}\varepsilon = \varepsilon' f_0$. We now take an arbitrary basis element x_1 of C_1; because

$$\varepsilon' f_0 \partial_1(x_1) = f\varepsilon\partial_1(x_1) = 0$$

and $\operatorname{im}\partial_1' = \ker\varepsilon'$, there exists $y_1' \in C_1'$ such that $\partial_1'(y_1') = f_0\partial_1(x_1)$. This defines a homomorphism $f_1\colon\ C_1 \to C_1'$ such that $f_0\partial_1 = \partial_1' f_1$.

Assume that we have inductively constructed the homomorphisms $f_i\colon\ C_i \to C_i'$ commuting with the boundary homomorphisms for $i \le n$; now, take the commutative diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} & \longrightarrow & \cdots \\
 & & & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\
\cdots & \longrightarrow & C'_{n+1} & \xrightarrow{\partial'_{n+1}} & C'_n & \xrightarrow{\partial'_n} & C'_{n-1} & \longrightarrow & \cdots
\end{array}$$

For every basis element x_{n+1} of C_{n+1}

$$\partial_n' f_n \partial_{n+1}(x_{n+1}) = f_{n-1}\partial_n\partial_{n+1}(x_{n+1}) = 0$$

that is to say, $f_n\partial_{n+1}(x_{n+1}) \in \ker\partial_n'$. It follows that $f_n\partial_{n+1}(x_{n+1})$ is an n-cycle of (C',∂'); but this chain complex is acyclic and so there exists $y_{n+1} \in C_{n+1}'$ such that $\partial'(y_{n+1}) = f_n\partial(x_{n+1})$. By extending linearly $x_{n+1} \mapsto y_{n+1}$, we obtain a homomorphism

$$f_{n+1}\colon\ C_{n+1} \to C_{n+1}'\ ,\ d' f_{n+1} = f_n d.$$

This concludes the inductive construction.

Suppose that $g\colon\ (C,\partial) \to (C',\partial')$ is another extension of $\bar f$. Then, for any arbitrary generator x_0 of C_0,

$$\varepsilon'(f_0 - g_0)(x_0) = 0;$$

since $\ker\varepsilon' = \operatorname{im}\partial_1'$, there exists an element $y_1 \in C_1'$ such that $\partial_1'(y) = (f_0 - g_0)(x_0)$. We define $s_0 : C_0 \to C_1'$ by $s_0(x_0) = y_1$ on the generators and extend this function linearly over the entire group C_0; in this way, we obtain a homomorphism $s_0 : C_0 \to C_1'$ such that $\partial_1' s_1 = f_0 - g_0$. Let us assume that, for every $i = 1,\cdots,n$, we have defined the homomorphisms $s_i : C_i \to C_{i+1}'$ satisfying the condition

$$\partial_{i+1}' s_i + s_{i-1}\partial_i = f_i - g_i.$$

For any generator x_{n+1} of C_{n+1}

$$\partial_{n+1}'(f_{n+1} - g_{n+1} - s_n\partial_{n+1})(x_{n+1}) = 0$$

(because $\partial_{n+1}' s_n + s_{n-1}\partial_n = f_n - g_n$); thus, there exists $y_{n+2} \in C_{n+2}'$ such that

$$(f_{n+1} - g_{n+1} - s_n\partial_{n+1})(x_{n+1}) = \partial_{n+2}'(y_{n+2}).$$

In this fashion, we construct a homomorphism $s_{n+1} : C_{n+1} \to C_{n+2}'$ and, in the end, we obtain a chain homotopy from f to g. ■

(II.3.7) Corollary. *Let (C,∂) and (C',∂') be two positive augmented chain complexes; assume (C,∂) to be free and (C',∂') to be acyclic. If $f\colon (C,\partial)\to(C',\partial')$ is an extension of the trivial homomorphism $0:\mathbb{Z}\to\mathbb{Z}$, then f is chain null-homotopic.*

The next result gives a good criterion to check if a positive, free, augmented chain complex is acyclic.

(II.3.8) Lemma. *Let (C,∂) be a positive, free, augmented chain complex with augmentation homomorphism*

$$\varepsilon\colon C_0 \longrightarrow \mathbb{Z}.$$

Then, (C,∂) is acyclic if and only if the following conditions hold true:

I. *There exists a function $\eta\colon \mathbb{Z}\to C_0$ such that $\varepsilon\eta=1$.*
II. *There exists a chain homotopy $s\colon C\to C$ such that*

1. *$\partial_1 s_0 = 1-\eta\varepsilon$,*
2. *$(\forall n\geq 1)\ \partial_{n+1}s_n + s_{n-1}\partial_n = 1$.*

Proof. Suppose that C is acyclic. Since $\varepsilon\colon C_0\to\mathbb{Z}$ is a surjection, there exists $x\in C_0$ such that $\varepsilon(x)=1$. Define

$$\eta\colon \mathbb{Z}\to C_0\,,\ n\mapsto nx.$$

Clearly $\varepsilon\eta=1$.

The homomorphisms $1\colon \mathbb{Z}\to\mathbb{Z}$ and $0\colon \mathbb{Z}\to\mathbb{Z}$ can be extended trivially to the chain homomorphisms $1,0\colon (C,\partial)\to(C,\partial)$; because of Theorem (II.3.6), there exists a chain homotopy $s\colon C\to C$ satisfying conditions 1 and 2.

Conversely, assume that there exists a chain homotopy $s\colon C\to C'$, and a homomrophism η with properties 1. and 2.; because of Noether's Homomorphism Theorem,

$$C_0/\ker\varepsilon\cong\mathbb{Z};$$

since $\varepsilon\partial_1=0$, we have that $\operatorname{im}\partial_1\subset\ker\varepsilon$ and, from $\partial_1 s_0=1-\eta\varepsilon$, we conclude that every $x\in C_0$ can be written as

$$x=\partial_1 s_0(x)+\eta\varepsilon(x).$$

Hence for every $x\in\ker\varepsilon$, we have that $x=\partial_1 s_0(x)$, that is to say, $\ker\varepsilon\subset\operatorname{im}\partial_1$, and therefore

$$H_0(C)\cong\mathbb{Z}.$$

Finally, because

$$\partial_{n+1}s_n+s_{n-1}\partial_n=1$$

in all positive dimensions, it follows that $H_n(C)=0$ for every $n>0$. Thus, (C,∂) is acyclic. ■

Theorem (II.3.6) and Corollary (II.3.7) require (C',∂') to be acyclic; this requirement can be replaced by a more interesting condition within the framework of the so-called *acyclic carriers*. The following definition is needed: given $(C,\partial),(C',\partial')\in\mathfrak{C}$, we say that (C',∂') is a *(chain) subcomplex* of (C,∂) if, for every $n\in\mathbb{Z}$, C'_n is a subgroup of C_n and $\partial'_n=\partial_n|C'_n$; we use the notation $(C',\partial')\le(C,\partial)$ to indicate that (C',∂') is a subcomplex of (C,∂). Let (C,∂) be a free chain complex; for each $n\in\mathbb{Z}$, let $\{x^{(n)}_\lambda\mid\lambda\in\Lambda_n\}$ be a basis of C_n. Now, let (C',∂') be an arbitrary chain complex. A *chain carrier* from (C,∂) to (C',∂') (relative to the choice of basis) is a function S, which associates with each basis element $x^{(n)}_\lambda$ a subcomplex $(S(x^{(n)}_\lambda),\partial_S)\le(C',\partial')$ satisfying the following properties:

1. $(S(x^{(n)}_\lambda),\partial_S)$ is an acyclic chain complex.
2. If x is a basis element of C_n such that $\partial x=\sum a_\lambda x^{(n-1)}_\lambda$ and $a_\lambda\neq 0$, then

$$(S(x^{(n-1)}_\lambda),\partial_S)\le(S(x),\partial_S).$$

We say that a morphism $f\in\mathfrak{C}((C,\partial),(C',\partial'))$ *has an acyclic carrier* S if $f(x^{(n)}_\lambda)\in S(x^{(n)}_\lambda)$ for every index λ and every $n\in\mathbb{Z}$. In this case, if x is a basis element, then $f(\partial(x))\in S(x)$. The next result is the **Acyclic Carrier Theorem**.

(II.3.9) Theorem. *Let $(C,\partial),(C',\partial')\in\mathfrak{C}$ be positive augmented chain complexes; suppose that (C,∂) is free and let S be an acyclic carrier from (C,∂) to (C',∂'). Then, any homomorphism $\bar{f}\colon\mathbb{Z}\to\mathbb{Z}$ has an extension $f\colon(C,\partial)\to(C',\partial')$ with chain carrier S. The chain homomorphism f is uniquely defined, up to chain homotopy.*

Proof. Take any generator x_0 of C_0; let $S(x_0)\le(C',\partial')$ be the acyclic subcomplex defined by S. Notice that the restriction of ε' to $S(x_0)_0$ is an augmentation homomorphism for $S(x_0)$. Since such a restriction is a surjection, there exists $y_0\in S(x_0)_0$ such that $\varepsilon'(y_0)=\bar{f}\varepsilon(x_0)$; the usual argument determines f_0 with carrier S.

We continue the proof using an induction procedure as in Theorem (II.3.6). Assume that, for every $i\le n$, we have constructed the homomorphisms $f_i: C_i\to C'_i$ that commute with the boundary homomorphisms. Notice that if x_{n+1} is an arbitrary generator of C_{n+1}

$$\partial'_n f_n\partial_{n+1}(x_{n+1})=f_{n-1}\partial_n\partial_{n+1}(x_{n+1})=0;$$

on the other hand, $f_n\partial_{n+1}(x_{n+1})$ belongs to the acyclic subcomplex

$$S(x_{n+1})\le(C',\partial')$$

and thus there exists $y_{n+1}\in C'_{n+1}\cap S(x_{n+1})$ such that $d'(y_{n+1})=f_nd(x_{n+1})$. The function $x_{n+1}\mapsto y_{n+1}$ can be linearly extended to the homomorphism

$$f_{n+1}\colon C_{n+1}\to C'_{n+1}$$

such that $d'f_{n+1}=f_nd$.

Also the proof of the second part follows the steps of the proof given in Theorem (II.3.6). ■

(II.3.10) Corollary. *Let (C,∂) and (C',∂') be given positive augmented chain complexes and let (C,∂) be free. Then any chain homomorphism $f\colon (C,\partial) \to (C',\partial')$, extending the trivial homomorphism $0\colon \mathbb{Z} \to \mathbb{Z}$ and having an acyclic carrier S, is null-homotopic.*

We prove now an important result known as **Five Lemma**.

(II.3.11) Lemma. *Let the diagram of Abelian groups and homomorphisms*

$$\begin{array}{ccccccccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & D & \xrightarrow{k} & E \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\varepsilon} \\
A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & D' & \xrightarrow{k'} & E'
\end{array}$$

be commutative and with exact lines. If the homomorphisms α, β, δ, and ε are isomorphisms, so is γ.

Proof. Let $c \in C$ be such that $\gamma(c) = 0$; then $\delta h(c) = h'\gamma(c) = 0$ and because δ is an isomorphism, $h(c) = 0$. In view of the exactness condition, there is a $b \in B$ with $g(b) = c$ and $g'\beta(b) = \gamma g(b) = 0$; thus, there exists $a' \in A'$ such that $f'(a') = \beta(b)$. But

$$c = g(b) = g\beta^{-1}f'(a') = gf\alpha^{-1}(a') = 0$$

and so γ is injective. For an arbitrary $c' \in C'$,

$$k\delta^{-1}h'(c') = \varepsilon^{-1}k'h'(c') = 0$$

and, hence, there exists $c \in C$ such that $h(c) = \delta^{-1}h'(c')$. Moreover,

$$h'(c' - \gamma(c)) = h'(c') - \delta\delta^{-1}h'(c') = 0$$

and hence, there exists $b' \in B'$ such that $g'(b') = c' - \gamma(c)$. It follows that

$$\gamma(c + g\beta^{-1}(b')) = \gamma(c) + g'\beta\beta^{-1}(b') = c'$$

and so γ is also surjective. ■

Exercises

1. A short exact sequence of Abelian groups

$$G_{n+1} \overset{f_{n+1}}{\rightarrowtail} G_n \overset{f_n}{\twoheadrightarrow} G_{n-1}$$

splits (or *is split*) if there exists a homomorphism $h_{n-1} : G_{n-1} \to G_n$ such that $f_n h_{n-1} = 1_{G_{n-1}}$ (or if there exists a homomorphism $k_n : G_n \to G_{n+1}$ such that $k_n f_{n+1} = 1_{G_{n+1}}$). Prove that if the short exact sequence

$$G_{n+1} \xrightarrow{f_{n+1}} G_n \xrightarrow{f_n} G_{n-1}$$

splits, then

$$G_n \cong G_{n+1} \oplus G_{n-1}.$$

2. Prove Theorem (II.3.3).

II.4 Simplicial Homology

In this section, we give some results which allow us to study more in depth the homology of a simplicial complex. We begin with some important remarks on the homology of a simplicial complex K. The groups $C_n(K)$ of the n-chains are free, with rank equal to the (finite) number of n-simplexes of K; hence, also the subgroups $Z_n(K)$ and $B_n(K)$ of $C_n(K)$ are free, with a finite number of generators. Finally, the homology groups $H_n(K)$ are Abelian and finitely generated; therefore, by the decomposition theorem for finitely generated Abelian groups, they are isomorphic to direct sums

$$\mathbb{Z}^{\beta(n)} \oplus \mathbb{Z}_{n(1)} \oplus \ldots \oplus \mathbb{Z}_{n(k)}$$

where $\mathbb{Z}_{n(i)}$ is cyclic of order $n(i)$. The number $\beta(n)$ – equal to the *rank* of the Abelian group $H_n(K)$ – is the nth-*Betti number* of the complex K.

Let p be the dimension of the simplicial complex K; for each $0 \le n \le p$, let $s(n)$ be the number of n-simplexes of K (remember that K is finite). Hence, the rank of the free Abelian group $C_n(K)$ is $s(n)$. We indicate with $z(n)$ and $b(n)$ the ranks of the groups $Z_n(K)$ and $B_n(K)$, respectively, where $n = 0, \ldots, p$. Since the boundary homomorphism $\partial_n \colon C_n(K) \to C_{n-1}(K)$ is a surjection on $B_{n-1}(K)$, by Nöther's Homomorphism Theorem, for each $n \ge 1$

$$(1) \qquad s(n) - z(n) = b(n-1);$$

if $n = 0$, we have $s(0) = z(0)$ because $C_0(K) = Z_0(K)$ and $B_{-1}(K) = 0$; on the other hand, $H_n(K,\mathbb{Z}) = Z_n(K)/B_n(K)$ and so

$$(2) \qquad \beta(n) = z(n) - b(n)$$

for $n \ge 0$. Subtracting (2) from (1) (when $n \ge 1$) it follows that

$$(3) \qquad s(n) - \beta(n) = b(n) - b(n-1).$$

If we do the alternate sum of the equalities (3) together with $s(0)-\beta(0)=b(0)$, we obtain

$$\sum_{n=0}^{p}(-1)^n(s(n)-\beta(n))=\pm\beta(p);$$

since $C_{p+1}(K)=0$, we have that $\beta(p)=0$ and so the equality

$$\sum_{n=0}^{p}(-1)^n s(n)=\sum_{n=0}^{p}(-1)^n\beta(n)$$

holds true. The number

$$\chi(K)=\sum_{n=0}^{p}(-1)^n\beta(n)$$

is the *Euler-Poincaré characteristic* of K; this may be useful in determining the homology of some finite simplicial complexes.

Let L be a simplicial subcomplex of a simplicial complex K; we now ask whether it is possible to compare the homology of a subcomplex $L\subset K$ with the homology of K. The (positive) answer lies with the *exact homology sequence* of the pair (K,L). Let us see how we may find this exact sequence. For every $n\geq 0$, consider the quotient of the chain groups $C_n(K)/C_n(L)$ and define

$$\partial_n^{K,L}\colon\ C_n(K)/C_n(L)\to C_{n-1}(K)/C_{n-1}(L)$$

by

$$\partial_n^{K,L}(c+C_n(L))=(\partial_n^K(c))+C_{n-1}(L).$$

This is a well-defined formula because, if c' is another representative of $c+C_n(L)$, then, $c-c'\in C_n(L)$ and

$$\partial_n^K(c-c')=\partial_n^L(c-c')\in C_{n-1}(L)\ ;$$

hence, $\partial_n^{K,L}(c+C_n(L))=\partial_n^{K,L}(c'+C_n(L))$. The reader can easily verify that the homomorphisms $\partial_n^{K,L}$ are boundary homomorphisms and so that

$$C(K,L)=\{C_n(K)/C_n(L),\partial_n^{K,L}\}$$

is a chain complex whose homology groups $H_n(K,L;\mathbb{Z})$ are the so-called *relative homology groups* of the pair (K,L). We point out that

$$H_n(K,L;\mathbb{Z})=Z_n(K,L)/B_n(K,L)$$

where

$$Z_n(K,L)=\ker\partial_n^{K,L}\text{ and }B_n(K,L)=\operatorname{im}\partial_{n+1}^{K,L}.$$

Let **CCsim** be the category whose objects are pairs (K,L), where K is a simplicial complex, L is one of its subcomplexes, and whose morphisms are pairs of simplicial functions

$$(k,\ell)\colon\ (K,L)\longrightarrow(K',L')$$

such that $k\colon\ K \to K'$ and $\ell\colon\ L \to L'$ is the restriction of k to L. The reader can easily verify that the relative homology determines a covariant functor

$$H(-,-;\mathbb{Z})\colon\ \mathbf{CCsim} \longrightarrow \mathbf{Ab}^{\mathbb{Z}}.$$

The next result, which is an immediate application of the Long Exact Sequence Theorem (II.3.1), is called **Long Exact Homology Sequence Theorem**; it relates the homology groups of L, K, and (K,L) to each other.

(II.4.1) Theorem. *Let (K,L) be a pair of simplicial complexes. For every $n > 0$, there is a homomorphism*

$$\lambda_n\colon\ H_n(K,L;\mathbb{Z}) \to H_{n-1}(L;\mathbb{Z})$$

(connecting homomorphism) that causes the following sequence of homology groups

$$\ldots \to H_n(L;\mathbb{Z}) \xrightarrow{H_n(i)} H_n(K;\mathbb{Z}) \xrightarrow{q_*(n)} H_n(K,L;\mathbb{Z}) \xrightarrow{\lambda_n} H_{n-1}(L;\mathbb{Z}) \to \ldots\ ,$$

to be exact; here, $H_n(i)$ is the homomorphism induced by the inclusion $i\colon\ L \to K$ and $q_*(n)$ is the homomorphism induced by the quotient homomorphism $q_n\colon\ C_n(K) \to C_n(K)/C_n(L)$.

Proof. For every $n > 0$, let

$$q_n\colon\ C_n(K) \to C_n(K)/C_n(L)$$

be the quotient homomorphism. With the given definitions, it is easily proved that

$$(\forall n \geq 0)\ \partial_n^{K,L} q_n = q_{n-1}\partial_n^K$$

and therefore,

$$q = \{q_n\}\colon\ C(K) \to C(K,L)$$

is a homomorphism of chain complexes. We note furthermore that for each $n \geq 0$, the sequence of Abelian groups

$$C_n(L) \xrightarrow{C_n(i)} C_n(K) \xrightarrow{\quad q_n \quad} C_n(K)/C_n(L)$$

is a short exact sequence and therefore, we have a short exact sequence of chain complexes

$$C(L) \xrightarrow{C(i)} C(K) \xrightarrow{\quad q \quad} C(K,L);$$

the result follows from Theorem (II.3.1). ∎

The exact sequence of homology groups described in the statement of Theorem (II.4.1) is the *exact homology sequence of the pair* (K,L).

In the context of the categories **Csim** and **CCsim**, the naturality of the connecting homomorphism

$$\lambda_n\colon\ H_n(K,L;\mathbb{Z}) \to H_{n-1}(L;\mathbb{Z})$$

can be explained as follows. We start with a result whose proof is easily obtained from the given definitions and is left to the reader.

(II.4.2) Theorem. *Let* $(k,\ell)\colon\ (K,L) \to (K',L')$ *be a given simplicial function. Then, for every* $n \geq 1$*, the following diagram commutes.*

$$\begin{array}{ccc} H_n(K,L;\mathbb{Z}) & \xrightarrow{\lambda_n} & H_{n-1}(L;\mathbb{Z}) \\ \downarrow{\scriptstyle H_n(k,\ell)} & & \downarrow{\scriptstyle H_{n-1}(\ell)} \\ H_n(K',L';\mathbb{Z}) & \xrightarrow[\lambda_n]{} & H_{n-1}(L';\mathbb{Z}) \end{array}$$

Let

$$pr_2\colon\ \mathbf{CCsim} \to \mathbf{Csim}$$

be the functor defined by

$$(\forall (K,L) \in \mathbf{CCsim})\ pr_2(K,L) = L$$

and

$$(\forall (k,\ell) \in \mathbf{CCsim}((K,L),(K',L')))\ pr_2(k,\ell) = \ell\ .$$

For each $n \geq 0$, take the covariant functors

$$H_n(-,-)\colon\ \mathbf{CCsim} \to Gr$$

and

$$H_{n-1}(-) \circ pr_2\colon\ \mathbf{CCsim} \to Gr\ .$$

Theorem (II.3.3) states that

$$\lambda_n\colon\ H_n(-,-;\mathbb{Z}) \to H_{n-1}(-;\mathbb{Z}) \circ pr_2$$

is a *natural transformation* (see the definition of natural transformation of functors in Sect. I.2).

Computing the homology of a complex K can be made easier by the exact homology sequence, provided that we can compute the homology of L and the relative homology of (K,L). Another very useful technique for computing the homology of a simplicial complex is using the *Mayer–Vietoris sequence*. Consider two simplicial complexes $K_1 = (X_1,\Phi_1)$ and $K_2 = (X_2,\Phi_2)$ such that $K_1 \cap K_2$ and $K_1 \cup K_2$ are simplicial complexes; in addition, $K_1 \cap K_2$ must be a subcomplex of both K_1 and K_2. The inclusions

$$\Phi_1 \cap \Phi_2 \hookrightarrow \Phi_\alpha\ ,\ \Phi_\alpha \hookrightarrow \Phi_1 \cup \Phi_2\ ,\ \alpha = 1,2$$

define simplicial functions

$$i_\alpha\colon\ K_1\cap K_2 \longrightarrow K_\alpha\ ,\ j_\alpha\colon\ K_\alpha \longrightarrow K_1\cup K_2\ ,\ \alpha=1,2$$

which, in turn, define the homomorphisms

$$\begin{aligned}\tilde{\imath}(n)\colon\ & C_n(K_1\cap K_2)\to C_n(K_1)\oplus C_n(K_2)\\ & c\mapsto (C_n(i_1)(c),C_n(i_2)(c))\ ,\end{aligned}$$

$$\begin{aligned}\tilde{\jmath}(n)\colon\ & C_n(K_1)\oplus C_n(K_2)\to C_n(K_1\cup K_2)\\ & (c,c')\mapsto C_n(j_1)(c)-C_n(j_2)(c').\end{aligned}$$

These homomorphisms have the following properties:

1. $\tilde{\imath}(n)$ is injective;
2. $\tilde{\jmath}(n)$ is surjective;
3. im $\tilde{\imath}(n)=\ker\,\tilde{\jmath}(n)$;
4. $(\partial_n^{K_1}\oplus\partial_n^{K_2})\tilde{\imath}(n)=\tilde{\imath}(n-1)\partial_n^{K_1\cap K_2}$;
5. $\tilde{\jmath}(n-1)(\partial_n^{K_1}\oplus\partial_n^{K_2})=\partial_n^{K_1\cup K_2}\tilde{\jmath}(n)$.

In this way, the chain complex sequence

$$0\to C(K_1\cap K_2)\xrightarrow{\ \tilde{\imath}\ } C(K_1)\oplus C(K_2)\xrightarrow{\ \tilde{\jmath}\ } C(K_1\cup K_2)\to 0$$

is short exact.

Theorem (II.3.1) enables us to state the next theorem, known as **Mayer–Vietoris Theorem**:

(II.4.3) Theorem. *For every $n\in\mathbb{Z}$, there is a homomorphism*

$$\lambda_n\colon\ H_n(K_1\cup K_2;\mathbb{Z})\longrightarrow H_{n-1}(K_1\cap K_2;\mathbb{Z})$$

such that the infinite sequence of homology groups

$$\begin{aligned}\ldots\to H_n(K_1\cap K_2;\mathbb{Z})&\xrightarrow{H_n(\tilde{\imath})} H_n(K_1;\mathbb{Z})\oplus H_n(K_2;\mathbb{Z})\\ &\xrightarrow{H_n(\tilde{\jmath})} H_n(K_1\cup K_2;\mathbb{Z})\xrightarrow{\lambda_n} H_{n-1}(K_1\cap K_2;\mathbb{Z})\to\ldots\end{aligned}$$

is exact.

We now give some results on the homology of certain simplicial complexes. Given a simplicial complex $K=(X,\Phi)$, we say that two vertices $x,y\in X$ are *connected* if there is a sequence of 1-simplexes

$$\{\{x_0^i,x_1^i\}\in\Phi, i=0,\ldots,n\}$$

where $x_0^0=x, x_1^n=y$, and $x_1^i=x_0^{i+1}$; we then have an equivalence relation on the set X, braking it down into a union of disjoint subsets $X=X_1\sqcup X_2\sqcup\ldots\sqcup X_k$. The sets

$$\Phi_i = \{\sigma \in \Phi | (\exists x \in X_i | x \in \sigma)\} \ , \ i = 1, \ldots, k$$

are disjoint; moreover, the pairs $K_i = (X_i, \Phi_i)$, $i = 1, \ldots, k$, called *connected components* of K, are simplicial subcomplexes of K. Hence, the relation of connectedness subdivides the complex K into a union of disjoint simplicial subcomplexes of K. From this point of view, a complex K is *connected* if and only if it has a unique connected component.

(II.4.4) Lemma. *A simplicial complex K is connected if and only if $|K|$ is connected.*

Proof. Suppose K to be connected and let p and q be any two points of $|K|$. Join p to a vertex x of its carrier $\underline{s(p)}$ by means of the segment with end points p and x; this segment is contained in $|s(p)|$ and is therefore a segment of $|K|$; similarly, join q to a vertex y of its carrier $s(q)$. However, the vertices x and y are also vertices of K and since K is connected, there is a path of 1-simplexes of K which links x to y. In this manner, we obtain a path of $|K|$ that links p to q; hence, $|K|$ is path-connected and so, $|K|$ is connected (see Theorem (I.1.21)).

Conversely, suppose $|K|$ to be connected and let K_i be a connected component of K; since K_i and $K \setminus K_i$ are subcomplexes of K, we have that $|K_i|$ is open and closed in $|K|$; since $|K|$ is connected, $|K_i| = |K|$, that is to say, $K_i = K$ and so, K is connected. ■

The reader is encouraged to review the results on connectedness and path-connectedness in Sect. I.1; note that these two concepts are equivalent for polyhedra.

(II.4.5) Lemma. *The following properties regarding a simplicial complex $K = (X, \Phi)$ are equivalent:*

1. *K is connected;*
2. *$H_0(K;\mathbb{Z}) \simeq \mathbb{Z}$;*
3. *the kernel of the augmentation homomorphism*

$$\varepsilon\colon\, C_0(K) \to \mathbb{Z} \ , \ \sum_{i=1}^{n} g_i\{x_i\} \mapsto \sum_{i=1}^{n} g_i$$

coincides with the group $B_0(K)$.

Proof. $1 \Rightarrow 3$: We first notice that the inclusion

$$B_0(K) \subset \ker\, \varepsilon$$

is always true: indeed,

$$\varepsilon\left(\partial_1\left(\sum_{i=0}^{k} g_i\{x_0^i, x_1^i\}\right)\right) = \sum_{i=0}^{k} g_i - \sum_{i=0}^{k} g_i = 0.$$

Let x be a fixed vertex of K. The connectedness of K means that, for every vertex y of K, the 0-cycles $\{x\}$ and $\{y\}$ are homological and so, for every 0-chain $c_0 = \sum_{i=0}^{k} g_i\{x_i\}$, there exists a 1-chain c_1 such that

$$\sum_{i=0}^{k} g_i\{x_i\} - \left(\sum_{i=0}^{k} g_i\right)\{x\} = \partial_1(c_1).$$

Therefore, it is clear that $c_0 \in \ker\,\varepsilon$ implies $c_0 \in B_0(K)$.

$3 \Rightarrow 2$: Given two homological 0-cycles z_0 and z_0', it follows from the property $z_0 - z_0' \in B_0(K)$ that $\varepsilon(z_0) = \varepsilon(z_0')$ and so we may define the homomorphism

$$\theta\colon\ H_0(K;\mathbb{Z}) \to \mathbb{Z}\ ,\ z_0 + B_0(K) \mapsto \varepsilon(z_0)$$

which is easily seen (by hypothesis 3) to be injective. The surjectivity of θ follows immediately; in fact, for every $g \in \mathbb{Z}$, we have $\theta(g\{x\} + B_0(K)) = g$, where $x \in X$ is a fixed vertex.

$2 \Rightarrow 1$: Let $K = K_1 \sqcup K_2 \sqcup \ldots \sqcup K_k$ be the decomposition of K into its connected components. We obtain

$$H_0(K;\mathbb{Z}) \simeq \sum_{i=1}^{k} H_0(K_i;\mathbb{Z}) \simeq \sum_{i=1}^{k} \mathbb{Z}$$

from the given definitions and from what we have proved so far; however, since $H_0(K;\mathbb{Z}) \simeq \mathbb{Z}$, we must have $k = 1$, which means that K is connected. ■

The next three examples are examples of abstract simplicial complexes called *acyclic* because they induce chain complexes which are acyclic (see Sect. II.3).

Homology of $\overline{\sigma}$ – Let $\overline{\sigma}$ be the simplicial complex generated by a simplex $\sigma = \{x_0, x_1, \ldots, x_n\}$. Since $\overline{\sigma}$ is connected, Lemma (II.4.5) ensures that $H_0(\overline{\sigma},\mathbb{Z}) = \mathbb{Z}$. We wish to prove that $H_i(\overline{\sigma},\mathbb{Z}) = 0$ for every $i > 0$. With this in mind, we begin to order the set of vertices, assuming that x_0 is the first element. Then, for any integer $0 < j < n$ and any ordered simplex $\{x_{i_0}, \ldots, x_{i_j}\}$, we define

$$k_j(\{x_{i_0}, \ldots, x_{i_j}\} = \begin{cases} \{x_0, x_{i_0}, \ldots, x_{i_j}\} & \text{for } i_0 > 0 \\ 0 & \text{for } i_0 = 0 \end{cases}$$

and linearly extend it to all j-chain of $\overline{\sigma}$ and therefore, to a homomorphism

$$k_j\colon\ C_j(\overline{\sigma}) \longrightarrow C_{j+1}(\overline{\sigma}).$$

A simple computation (on the simplexes of $\overline{\sigma}$) shows that for every chain $c \in C_j(\overline{\sigma})$

$$\partial_{j+1}k_j(c) + k_{j-1}\partial_j(c) = c$$

and so any $z_j \in Z_j(\overline{\sigma})$ is a boundary, that is to say, $H_j(\overline{\sigma},\mathbb{Z}) \cong 0$. Regarding $H_n(\overline{\sigma},\mathbb{Z})$, we note that σ, being the only n-simplex of $\overline{\sigma}$, cannot be a cycle; consequently, $Z_n(\overline{\sigma}) \cong 0$.

Homology of a simplicial cone – Since $\sigma = \{x_0, x_1, \ldots, x_{n+1}\}$, we call the simplicial complex

$$C(\sigma) = \dot{\sigma} \smallsetminus \{x_1, \ldots, x_{n+1}\},$$

obtained by removing the n-face opposite to the vertex x_0 from the simplicial complex $\dot{\sigma}$, *n-simplicial cone with vertex* $\{x_0\}$. Clearly

$$H_0(C(\sigma),\mathbb{Z}) \cong \mathbb{Z}$$

because $C(\sigma)$ is connected. A similar proof to the one used for $\overline{\sigma}$ shows that $H_j(C(\sigma),\mathbb{Z}) \cong 0$ for every $0 < j < n$. We note that, when $j = n$, the vertex x_0 belongs to every n-simplex of $C(\sigma)$ and so

$$(\forall c \in C_n(C(\sigma)))c = k_{n-1}\partial_n(c),$$

allowing us to conclude that the trivial cycle 0 is the only n-cycle of $C(\sigma)$; in other words, $H_n(C(\sigma);\mathbb{Z}) \cong 0$.

In the next example we refer to the construction of an acyclic carrier.

Homology of the (abstract) cone – Let $vK = v * K$ be the join of a simplicial complex $K = (X,\Phi)$ and of a simplicial complex with a single vertex (and simplex) v.

(II.4.6) Lemma. *The cones vK are acyclic simplicial complexes.*

Proof. Let v be the simplicial complex with the single vertex v and no other simplex; it is clear that v (considered as a simplicial complex) is an acyclic simplicial complex. The chain complex $C(v)$ is a subcomplex of $C(vK)$; let $\iota\colon\ C(v) \to C(vK)$ be the inclusion. Consider the simplicial function

$$c\colon\ vK \to v\ ,\ y \in v\Phi \mapsto \{v\}\ .$$

It is readily seen that the chain morphism

$$C(c)\iota\colon\ C(v) \longrightarrow C(v)$$

coincides with the identity homomorphism of $C(v)$; then, for every $n \in \mathbb{Z}$, the composite $H_n(c)H_n(\iota)$ equals the identity. Let us prove that $\iota C(c)$ and the identity homomorphism $1_{C(vK)}$ of $C(vK)$ are homotopic. We define

$$s_n\colon\ C_n(vK) \to C_{n+1}(vK)$$

on the oriented n-simplexes $\sigma \in v\Phi$ (understood as a chain) by the formula

$$s_n(\sigma) = \begin{cases} 0 & \text{if } v \in \sigma, \\ v\sigma & \text{if } v \notin \sigma; \end{cases}$$

s_n may be linearly extended to a homomorphism of $C_n(vK)$. Let us take a look into the properties of these functions.

Case 1: $n = 0$ – Let x be any vertex of vK.

$$(1_{C_0(vK)} - \iota C_0(c))(x) = \begin{cases} x - v & \text{if } x \neq v, \\ 0 & \text{if } x = v. \end{cases}$$

$$\partial_1 s_0(x) = \begin{cases} x - v & \text{if } x \neq v, \\ 0 & \text{if } x = v. \end{cases}$$

Case 2: $n > 0$ – Let σ be any oriented n-simplex of vK. We first observe that $\iota C_n(c)(\sigma) = 0$; moreover, if v is not a vertex of σ, we have

$$\partial_{n+1}(v\sigma) = \sigma - v\partial_n(\sigma).$$

Consequently, $v \notin \sigma$ implies

$$s_{n-1}\partial_n(\sigma) + \partial_{n+1}s_n(\sigma) = v\partial_n(\sigma) + \partial_{n+1}(v\sigma) = \sigma.$$

We now suppose that $v \in \sigma$. Then,

$$s_{n-1}\partial_n(\sigma) + \partial_{n+1}s_n(\sigma) = s_{n-1}\partial_n(\sigma) = \sigma.$$

It follows from these remarks that $\iota C(c)$ and the identity homomorphism $1_{C(vK)}$ are homotopic and we conclude from Proposition (II.3.4) that, for every $n \in \mathbb{Z}$, $H_n(\iota)H_n(c)$ coincides with the identity homomorphism. ■

We now seek a better understanding of the relative homology $H_*(K,L;\mathbb{Z})$ of a pair of simplicial complexes (K,L). As usual, $K = (X,\Phi)$ and $L = (Y,\Psi)$ with $Y \subset X$ and $\Psi \subset \Phi$. Let v be a point which is not in the set of vertices of either K or L. Let CL be the abstract cone vL. It follows from the definitions that $K \cap CL = L$.

(II.4.7) Theorem. *The homology groups $H_n(K,L;\mathbb{Z})$ and $H_n(K \cup CL;\mathbb{Z})$ are isomorphic for each $n \geq 1$.*

Proof. The central idea in this proof is to compare the exact homology sequence of the pair (K,L) and the exact sequence of Mayer–Vietoris of K and CL, before using the Five Lemma; the notation is the one already adopted for the Mayer–Vietoris Theorem.

Let us consider the simplicial function $f\colon K \to CL$ defined on the vertices by

$$f(x) = \begin{cases} x & \text{if } x \in Y \\ v & \text{if } x \in X \setminus Y. \end{cases}$$

For each nonnegative integer n, we now define the homomorphisms

$$\tilde{k}_n\colon C_n(K) \to C_n(K) \oplus C_n(CL)\ ,\ c \mapsto (c, C_n(f)(c))$$

and

$$\tilde{h}_n\colon C_n(K)/C_n(L) \longrightarrow C_n(K \cup CL),$$
$$c + C_n(L) \mapsto C_n(j_1)(c) - C_n(j_2)C_n(f)(c)$$

(that is to say, $\tilde{h}_n(c + C_n(L)) = \tilde{\jmath}_n\tilde{k}_n(c)$). The function $\tilde{h}_n$ is well defined; in fact, had $c' \in C_n(K)$ been such that $c - c' \in C_n(L)$, we would have $c - c' \in C_n(K \cap CL)$ and so

$$\tilde{\jmath}_n\tilde{k}_n(c - c') = \tilde{\jmath}_n\tilde{\imath}_n(c - c') = 0.$$

The homomorphism sequences $\tilde{h} = \{\tilde{h}_n | n \geq 0\}$ and $\tilde{k} = \{\tilde{k}_n | n \geq 0\}$ are homomorphisms of chain complexes giving rise to a commutative diagram

$$\begin{array}{ccccc}
C(L) & \stackrel{C(i)}{\rightarrowtail} & C(K) & \stackrel{\tilde{q}}{\twoheadrightarrow} & C(K,L) \\
\downarrow 1 & & \downarrow \tilde{k} & & \downarrow \tilde{h} \\
C(K \cap CL) & \stackrel{\tilde{\imath}}{\rightarrowtail} & C(K) \oplus C(CL) & \stackrel{\tilde{\jmath}}{\twoheadrightarrow} & C(K \cup CL)
\end{array}$$

Since CL is an acyclic simplicial complex, we obtain, for every $n \geq 2$, the commutative diagram of Abelian groups

$$\begin{array}{ccccccccc}
H_n(L;\mathbb{Z}) & \longrightarrow & H_n(K;\mathbb{Z}) & \longrightarrow & H_n(K,L;\mathbb{Z}) & \longrightarrow & H_{n-1}(L;\mathbb{Z}) & \longrightarrow & H_{n-1}(K;\mathbb{Z}) \\
1\downarrow & & \cong\downarrow & & \gamma\downarrow & & 1\downarrow & & \cong\downarrow \\
H_n(L;\mathbb{Z}) & \longrightarrow & H_n(K;\mathbb{Z}) & \longrightarrow & H_n(K \cup CL;\mathbb{Z}) & \longrightarrow & H_{n-1}(L;\mathbb{Z}) & \longrightarrow & H_{n-1}(K;\mathbb{Z})
\end{array}$$

and by the Five Lemma, we conclude that γ is an isomorphism; when $n = 1$, the last vertical arrow is an injective homomorphism

$$H_0(K;\mathbb{Z}) \longrightarrow H_0(K;\mathbb{Z}) \oplus \mathbb{Z}$$

and again with an argument similar to the Five Lemma, we conclude that γ is an isomorphism. ■

(II.4.8) Remark. We recall that we have defined the relative homology groups of a pair of simplicial complexes (K,L) through the chain complex $C(K,L) = \{C_n(K)/C_n(L), \partial_n^{K,L}\}$; we now construct the relative groups $H_n(K,L;\mathbb{Z})$, $n \geq 0$, from a slightly different point of view which turns out to be very useful for computing homology groups.

For any $n \geq 0$, let $\overline{C}_n(K,L)$ be the Abelian group of formal linear combinations, with coefficients in $\mathbb{Z}$, of all n-simplexes of K which are not in L; in other words, if $K = (X, \Phi)$, $L = (Y, \Psi)$ with $Y \subset X$ and $\Psi \subset \Phi$,

$$\overline{C}_n(K,L;\mathbb{Z}) = \{\sum_i m_i\sigma_n^i | \sigma_n^i \in \Phi \setminus \Psi\}.$$

The inclusion $i\colon L \to K$ induces an injective homomorphism $C_n(i)\colon C_n(L) \to C_n(K)$ for each $n \geq 0$; we now take, for every $n \geq 0$, the following linear homomorphisms:

$$\beta_n\colon C_n(K) \to C_n(L)$$

defined on the n-simplexes of K by the conditions

$$\beta_n(\sigma_n) = \begin{cases} 0 & \text{if } \sigma_n \in \Phi \setminus \Psi \\ \sigma_n & \text{if } \sigma_n \in \Psi \end{cases}$$

$$\alpha_n \colon \overline{C}_n(K,L) \to C_n(K)\ ,\ \sigma_n \in \Phi \setminus \Psi \mapsto \sigma_n$$

$$\mu_n \colon C_n(K) \to \overline{C}_n(K,L)$$

such that

$$\mu_n(\sigma_n) = \begin{cases} \sigma_n & \text{if } \sigma_n \in \Phi \setminus \Psi \\ 0 & \text{if } \sigma_n \in \Psi\ . \end{cases}$$

It is easy to check that $\beta_n C_n(i) = 1$, $\mu_n \alpha_n = 1$, $\mu_n C_n(i) = 0$, and $C_n(i)\beta_n + \alpha_n \mu_n = 1$ for each $n \geq 0$. Hence, for every $n \geq 0$, we have a short exact sequence

$$C_n(L) \xrightarrow{C_n(i)} C_n(K) \xrightarrow{\mu_n} \overline{C}_n(K,L).$$

We now consider the boundary homomorphism $\partial_n \colon C_n(K) \to C_{n-1}(K)$ and define

$$\overline{\partial}_n \colon \overline{C}_n(K,L) \to \overline{C}_{n-1}(K,L)$$

as the composite homomorphism $\overline{\partial}_n = \mu_{n-1}\partial_n\alpha_n$. We note that

$$\begin{aligned} \overline{\partial}_{n-1}\overline{\partial}_n &= (\mu_{n-2}\partial_{n-1}\alpha_{n-1})(\mu_{n-1}\partial_n\alpha_n) \\ &= (\mu_{n-2}\partial_{n-1})(1 - C_{n-1}(i)\beta_{n-1})\partial_n\alpha_n \\ &= \mu_{n-2}\partial_{n-1}\partial_n\alpha_n - \mu_{n-2}\partial_{n-1}C_{n-1}\beta_{n-1} = 0 \end{aligned}$$

since the factor $\partial_{n-1}\partial_n = 0$ appears in the first term and also because the second term is null on all $(n-1)$-simplex of K. The graded Abelian group $\{\overline{C}_n(K,L) \mid n \in \mathbb{Z}\}$, where $\overline{C}_n(K,L) = 0$ for every $n < 0$, has a boundary homomorphism $\{\overline{\partial}_n \mid n \in \mathbb{Z}\}$ with $\overline{\partial}_n = 0$ for $n \leq 0$; let

$$\overline{H}_*(K,L;\mathbb{Z}) = \{\overline{H}_n(K,L;\mathbb{Z})\}$$

be its homology. Let $\theta_n \colon \overline{C}_n(K,L) \to C_n(K)$ be the linear homomorphism defined on an n-simplex $\sigma_n \in \Phi \setminus \Psi$ by $\theta_n(\sigma_n) = \sigma_n + C_n(L)$ (if $n < 0$, we define $\theta_n = 0$). We note that θ_n commutes with the boundary homomorphisms; it is sufficient to verify this statement for an n-simplex $\sigma_n \in \Phi \setminus \Psi$:

$$\partial_n^{K,L}\theta_n(\sigma_n) = \partial_n(\sigma_n) + C_{n-1}(L) = \sum_{\sigma_{n-1,i} \in \Phi \setminus \Phi} (-1)^i \sigma_{n-1,i} + C_n(L);$$

$$\theta_{n-1}\overline{\partial}_n(\sigma_n) = \theta_{n-1}(\mu_{n-1}\sum_i (-1)^i \sigma_{n-1,i}) = \sum_{\sigma_{i,n-1} \in \Phi \setminus \Phi} (-1)^i \sigma_{n-1,i} + C_n(L).$$

Therefore, the set $\{\theta_n \mid n \in \mathbb{Z}\}$ induces a homomorphism

$$H_n(\theta_n)\colon\ \overline{H}_n(K,L;\mathbb{Z}) \to H_n(K,L;\mathbb{Z})\ .$$

On the other hand, θ_n is an isomorphism for each $n \geq 0$ (it is injective by definition and surjective because $\theta_n \mu_n = q_n$). Therefore, the two types of homology groups are isomorphic.

Let

$$\{K_i = (X_i, \Phi_i) \mid i = 1, \ldots, p\}$$

be a finite set of simplicial complexes; we choose a base vertex $x_0^i \in X_i$ for each K_i and construct the *wedge sum* of all K_i as the simplicial complex

$$\vee_{i=1}^{p} K_i\colon\ = \bigcup_{i=1}^{n} (\{x_0^1\} \times \ldots \times K_i \times \ldots \times \{x_0^p\})$$

that is to say

$$\vee_{i=1}^{p} K_i = (\vee_{i=1}^{p} X_i, \vee_{i=1}^{p} \Phi_i)\,.$$

The next theorem shows that the homology of the wedge sum of simplicial complexes acts in a special way.

(II.4.9) Theorem. *For every $q \geq 1$,*

$$H_q(\vee_{i=1}^{p} K_i;\mathbb{Z}) \cong \oplus_{i=1}^{p} H_q(K_i;\mathbb{Z}).$$

Proof. It is enough to prove this result for $p = 2$. The short exact sequence of chain complexes

$$C(K_1) \overset{i}{\rightarrowtail} C(K_1 \vee K_2) \overset{k}{\twoheadrightarrow} \overline{C}(K_1 \vee K_2, K_1;\mathbb{Z})$$

induces a long exact sequence of homology groups

$$\ldots \to H_n(K_1;\mathbb{Z}) \xrightarrow{H_n(i)} H_n(K_1 \vee K_2;\mathbb{Z}) \xrightarrow{q_*(n)} H_n(K_2;\mathbb{Z}) \xrightarrow{\lambda_n} H_{n-1}(K_1;\mathbb{Z}) \to \ldots$$

(see Remark (II.4.8)). Let us now examine how the homomorphisms of chain complexes

$$C(K_1) \xrightarrow{i} C(K_1 \vee K_2)$$

$$C(K_1 \vee K_2) \xrightarrow{k} \overline{C}(K_1 \vee K_2, K_1;\mathbb{Z}) \cong C(K_2)$$

are defined on simplexes (in other words, the generators of the free groups that concern us):

$$(\forall \sigma_n^1 \in \Phi_1)\ i_n(\sigma_n) = \sigma_n^1 \times \{x_0^2\}$$
$$(\forall \sigma_n^1 \times \{x_0^2\} \in \Phi_1 \times \{x_0^2\})\ k_n(\sigma_n^1 \times \{x_0^2\} = 0$$
$$(\forall \{x_0^1\} \times \sigma_n^2 \in \{x_0^1\} \times \Phi_2)\ k_n(\{x_0^1\} \times \sigma_n^2) = \sigma_n^2.$$

We now define the homomorphisms

$$j\colon\ C(K_1 \vee K_2) \to C(K_1) \text{ and } h\colon\ \overline{C}(K_1 \vee K_2, K_1; \mathbb{Z}) \to C(K_1 \vee K_2)$$

as follows:

$$(\forall \sigma_n^1 \times \{x_0^2\} \in \Phi_1 \times \{x_0^2\})\ j_n(\sigma_n^1 \times \{x_0^2\}) = \sigma_n^1$$
$$(\forall \{x_0^1\} \times \sigma_n^2 \in \{x_0^1\} \times \Phi_2)\ j_n(\{x_0^1\} \times \sigma_n^2) = 0$$
$$(\forall \sigma_n^2 \in \Phi_2)\ h_n(\sigma_n^2) = \{x_0^1\} \times \sigma_n^2.$$

Morphisms i, k, j, and h are induced by simplicial functions and so they commute with boundary operators. Moreover, $ji = 1_{C(K_1)}$ and $kh = 1_{C(K_2)}$, a property that extends to the respective homomorphisms regarding homology groups. Hence, for each $q \geq 1$, we have a splitting short exact sequence of homology groups

$$H_q(K_1; \mathbb{Z}) \xrightarrow{H_q(i)} H_q(K_1 \vee K_2; \mathbb{Z}) \xrightarrow{H_q(k)} H_q(K_2; \mathbb{Z}).$$ ■

II.4.1 Reduced Homology

It is sometimes an advantage to introduce a little change to the simplicial homology, named *reduced homology*; the only difference between the two homologies lies on the group $H_0(-; \mathbb{Z})$. To obtain the reduced homology $\widetilde{H}_*(K; \mathbb{Z})$ of a simplicial complex K, we consider the chain complex

$$\widetilde{C}(K, \mathbb{Z}) = \{\widetilde{C}_n(K), \widetilde{d}_n\}$$

where

$$\widetilde{C}_n(K) = \begin{cases} C_n(K)\ , & n \geq 0 \\ \mathbb{Z}\ , & n = -1 \\ 0\ , & n \leq -2 \end{cases}$$

and define the boundary homomorphism

$$\widetilde{d}_n = \begin{cases} \partial_n\ , & n \geq 1 \\ \varepsilon\colon\ C_0(K) \to \mathbb{Z}\ , & n = 0 \\ 0\ , & n \leq -1 \end{cases}$$

where ε is the augmentation homomorphism (see Lemma (II.4.5)). We only need to verify that $\widetilde{d}_0 \widetilde{d}_1 = 0$; but this follows directly from the definition of ε.

We leave to the reader, as an exercise, to prove that if K is a connected simplicial complex, then

$$(\forall n \neq 0)\ \widetilde{H}_n(K; \mathbb{Z}) \cong H_n(K; \mathbb{Z})$$

and

$$\widetilde{H}_0(K; \mathbb{Z}) \cong 0.$$

Exercises

1. The *simplicial n-sphere* is the simplicial complex

$$\dot{\sigma}_{n+1} = (\sigma_{n+1}, \Phi)$$

where $\sigma_{n+1} = \{x_0, x_1, \ldots, x_{n+1}\}$ and $\Phi = \wp(X) \setminus \{\emptyset, \sigma_{n+1}\}$. Prove that

$$H_p(\dot{\sigma}_{n+1}) = \begin{cases} \mathbb{Z}\,, & p = 0, n \\ 0\,, & p \neq 0, n. \end{cases}$$

2. Prove that a subgroup of a free Abelian group is free (if this proves to be very difficult, refer to [17], Theorem 5.3.1f).

3. Compute the homology groups of the triangulations associated with the following spaces (see Exercise 4 on p. 64).

a) cylinder $C = S^1 \times I$;
b) Möbius band M;
c) Klein bottle K;
d) real projective plane $\mathbb{R}P^2$;
e) G_2, obtained by adding two handles to the sphere S^2.

4. Compute the Betti numbers and the Euler–Poincaré characteristic for the surfaces of the previous exercise.

5. Let K be a given connected simplicial complex and $\Sigma K = K * \{x, y\}$ be the suspension of K (see examples of simplicial complexes given in Sect. II.2). Prove that

$$(\forall n \geq 0)\ \widetilde{H}_n(\Sigma K) \cong \widetilde{H}_{n-1}(K)$$

by means of the Mayer–Vietoris sequence.

6. Let K be a one-dimensional connected simplicial complex (namely, a graph), and $C(K) = 1 - \chi(K)$ its *cyclomatic number* (also called the *circuit rank*). Prove that $C(K) \geq 0$ and that the equality holds if and only if $|K|$ is contractible (that is to say, K is a *tree*).

II.5 Homology with Coefficients

In Sect. II.4, we have studied the homology of oriented simplicial complexes K determined by the chain complex

$$(C(K), \partial) = \{C_n(K), \partial_n^K | n \in \mathbb{Z}\},$$

$C_n(K)$ being the free Abelian groups of formal linear combinations with coefficients in $\mathbb{Z}$ of the n-simplexes of K. We now wish to generalize our homology with coefficients in the Abelian group $\mathbb{Z}$ to a homology with coefficients drawn from any Abelian group G.

We begin by reviewing the construction of the *tensor product* of two Abelian groups A, B: by definition, $A \otimes B$ is the Abelian group generated by the set of elements

$$\{a \otimes b | a \in A\ ,\ b \in B\}$$

where $(\forall a, a' \in A\ ,\ b, b' \in B)$

1. $(a + a') \otimes b = a \otimes b + a' \otimes b$,
2. $a \otimes (b + b') = a \otimes b + a \otimes b'$.

We notice that the function

$$A \otimes \mathbb{Z} \to A\ ,\ a \otimes n \mapsto na$$

is a group isomorphism, that is to say, $A \otimes \mathbb{Z} \cong A$ (similarly, $\mathbb{Z} \otimes A \cong A$). The reader may easily prove that

$$(A \oplus B) \otimes C \cong (A \otimes C) \oplus (B \otimes C)$$

for any three Abelian groups A, B, and C. Finally, given two group homomorphisms $\phi\colon\ A \to A'$ and $\psi\colon\ B \to B'$, the function $\phi \otimes \psi\colon\ A \otimes B \to A' \otimes B'$ defined by $\phi \otimes \psi(a \otimes b) = \phi(a) \otimes \psi(b)$ is a homomorphism of Abelian groups.

In this way, by fixing an Abelian group G we are able to construct a covariant functor

$$- \otimes G\colon\ \mathbf{Ab} \to \mathbf{Ab}$$

that transforms a group A into $A \otimes G$ and a morphism $\phi\colon\ A \to B$ into the morphism $\phi \otimes 1_G$.

We extend this functor to chain complexes. We transform a given chain complex $(C, \partial) \in \mathfrak{C}$ in $(C \otimes G, \partial \otimes 1_G)$, by setting

$$(C \otimes G)_n := C_n \otimes G$$

for every $n \in \mathbb{Z}$, and by defining the homomorphisms

$$(\partial \otimes 1_G)_n := \partial_n \otimes 1_G\colon\ C_n \otimes G \to C_{n-1} \otimes G\ .$$

Since

$$(\partial \otimes 1_G)_{n-1}(\partial \otimes 1_G)_n = (\partial_{n-1} \otimes 1_G)(\partial_n \otimes 1_G) = \partial_{n-1}\partial_n \otimes 1_G = 0\ ,$$

we conclude that $(C \otimes G, \partial \otimes 1_G)$ is a chain complex whose homology groups are the *homology groups of* (C, ∂) *with coefficients in* G. The nth-homology group of (C, ∂) with coefficients in G is defined by the quotient group

$$H_n(C;G) = \ker(\partial_n \otimes 1_G)/\operatorname{im}(\partial_{n+1} \otimes 1_G);$$

the graded Abelian group $H_*(C;G)$ is the graded *homology* group of C with coefficients in G. In particular, if $(C,\partial) = (C(K),\partial)$, the chain complex of the oriented complex K, then $H_*(C(K);G)$ – simply denoted by $H_*(K;G)$ – is the homology of K with coefficients in G.

We recall that the chain complex $(C(K),\partial)$ is positive, free, and has an augmentation homomorphism $\varepsilon\colon C_0(K) \to \mathbb{Z}$. To continue with our work, we only need one of these properties, namely, that (C,∂) be *free*.

For every free chain complex (C,∂) and for each $n \in \mathbb{Z}$, we have a short exact sequence of free Abelian groups

$$Z_n(C) \rightarrowtail C_n \xrightarrow{\partial_n} B_{n-1}(C)\ .$$

The main point is that, by taking the tensor product of each component of this exact sequence with G, we obtain again a short exact sequence.

(II.5.1) Lemma. *If*

$$A \overset{f}{\rightarrowtail} B \overset{g}{\twoheadrightarrow} C$$

is a short exact sequence of free Abelian groups and G is an Abelian group, then also the sequence

$$A \otimes G \overset{f\otimes 1_G}{\rightarrowtail} B \otimes G \overset{g \otimes 1_G}{\twoheadrightarrow} C \otimes G$$

is exact.

Proof. We begin by noting that the group C is free; therefore, we may define a map $s\colon C \to B$ simply by choosing for each element of a basis of C an element of its anti-image under g, and by extending this operation linearly; through this procedure, we obtain a homomorphism of Abelian groups

$$s\colon C \longrightarrow B$$

such that $gs = 1_C$, the identity homomorphism of C onto itself. It follows that $g(1_B - sg) = g - (gs)g = 0$, in other words, the image of $1_B - sg$ is contained in the $\ker g = \operatorname{im} f$; for this reason, we may define the map $r := f^{-1}(1_B - sg)\colon B \to A$ that also satisfies $rf = f^{-1}(1_B - sg)f = 1_A$. We thus obtain the relations

$$rf = 1_A\ , \quad gs = 1_C\ , \quad \text{and} \quad fr + sg = 1_B.$$

We know that the tensor product by G is a functor and that it transforms sums of homomorphisms into sums of transformed homomorphisms; consequently, tensorization gives us the relations

$$(r\otimes 1_G)(f\otimes 1_G)=1_{A\otimes G}\quad (g\otimes 1_G)(s\otimes 1_G)=1_{C\otimes G}$$

and

$$(f\otimes 1_G)(r\otimes 1_G)+(s\otimes 1_G)(g\otimes 1_G)=1_{B\otimes G}.$$

The first of these relations tells us that $(f\otimes 1_G)$ is injective; the second, that $(g\otimes 1_G)$ is surjective, and the third, that $\mathrm{im}(f\otimes 1_G)=\ker(g\otimes 1_G)$ because, if we take $x\in B\otimes G$ such that $(g\otimes 1_G)(x)=0$, then

$$\begin{aligned}x&=(f\otimes 1_G)(r\otimes 1_G)(x)+(s\otimes 1_G)(g\otimes 1_G)(x)\\&=(f\otimes 1_G)((r\otimes 1_G)(x))\in \mathrm{im}(f\otimes 1_G).\end{aligned}$$

■

Returning to our free chain complex (C,∂), we notice that for each integer n, the sequence

$$Z_n(C)\otimes G \rightarrowtail C_n\otimes G \xrightarrow{\partial_n\otimes 1_G} B_{n-1}(C)\otimes G$$

is short exact. In addition, we observe that the graded Abelian groups $Z(C)=\{Z_n(C)|n\in\mathbb{Z}\}$ and $B(C)=\{B_n(C)|n\in\mathbb{Z}\}$ may be viewed as chain complexes with trivial boundary operator 0; we then construct the chain complexes

1. $(Z(C)\otimes G,0\otimes 1_G)$;
2. $(C\otimes G,\partial\otimes 1_G)$;
3. $(\widetilde{B(C)}\otimes G,0\otimes 1_G)$, where $\widetilde{B(C)}_n=B_{n-1}(C)$

and observe that in view of the preceding short exact sequence of Abelian groups, we have a short exact sequence of chain complexes

$$(Z(C)\otimes G,0\otimes 1_G)\rightarrowtail (C\otimes G,\partial\otimes 1_G)\twoheadrightarrow (\widetilde{B(C)}\otimes G,0\otimes 1_G).$$

By the Long Exact Sequence Theorem (II.3.1), we obtain the long exact sequence of homology groups

$$\cdots\longrightarrow H_n(Z(C)\otimes G)\longrightarrow H_n(C\otimes G)\longrightarrow$$

$$H_n(\widetilde{B(C)}\otimes G)\longrightarrow H_{n-1}(Z(C)\otimes G)\longrightarrow\cdots$$

in other words, by considering the format of the boundary operators, we have the following exact sequence of Abelian groups:

$$\cdots\longrightarrow B_n(C)\otimes G\xrightarrow{i_n\otimes 1_G} Z_n(C)\otimes G\xrightarrow{j_n}$$

$$H_n(C;G)\xrightarrow{h_n} B_{n-1}(C)\otimes G\xrightarrow{i_{n-1}\otimes 1_G} Z_{n-1}(C)\otimes G\longrightarrow\cdots$$

Note that i_n is the inclusion of $B_n(C)$ in $Z_n(C)$ and j_n is the induced homomorphism by the inclusion of $Z_n(C)$ in C_n; the reader is also asked to notice that the connecting homomorphism λ_{n+1} in Theorem (II.3.1) coincides with $i_n\otimes 1_G$.

Since $\mathrm{im}\, j_n = \ker h_n$, we conclude that, for every $n \geq 0$, the sequence

$$\mathrm{im}\, j_n \rightarrowtail H_n(C;G) \xrightarrow{h_n} \mathrm{im}\, h_n$$

is short exact.

(II.5.2) Lemma. *If the group G is free, the short exact sequence*

$$\mathrm{im}\, j_n \rightarrowtail H_n(C;G) \xrightarrow{h_n} \mathrm{im}\, h_n$$

splits,[4] *and so*

$$H_n(C;G) \cong \mathrm{im}\, j_n \oplus \mathrm{im}\, h_n$$

(however, one should note that this isomorphism is not canonic).

Proof. Let us take the homomorphism of Abelian groups

$$h_n\colon\ H_n(C;G) \to B_{n-1}(C) \otimes G.$$

As a subgroup of the free Abelian group C_{n-1}, $B_{n-1}(C)$ is free and by hypothesis G is also free; then $B_{n-1}(C) \otimes G$ is free and it follows that $\mathrm{im}\, h_n$ is free. We now choose, for every generator $x \in \mathrm{im}\, h_n$, an element $y \in H_n(C;G)$ such that $h_n(y) = x$; by linearity, we obtain a homomorphism

$$s\colon\ \mathrm{im}\, h_n \longrightarrow H_n(C;G)$$

such that $h_n s = 1_{\mathrm{im}\, h_n}$. Exercise 1 in Sect. II.3 completes the proof.

The homomorphism s depends on the choice of the elements y for the generators x; therefore, s is not canonically determined. ■

We now give another interpretation of the groups $\mathrm{im}\, j_n$ and $\mathrm{im}\, h_n$. Note that

$$\mathrm{im}\, j_n \cong Z_n(C) \otimes G / \ker j_n = Z_n(C) \otimes G / \mathrm{im}(i_n \otimes 1_G);$$

the quotient group

$$Z_n(C) \otimes G / \mathrm{im}(i_n \otimes 1_G) := \mathrm{coker}(i_n \otimes 1_G)$$

is called *cokernel* of $i_n \otimes 1_G$. Since $\mathrm{im}\, h_n = \ker(i_{n-1} \otimes 1_G)$, the exact sequence

$$\mathrm{im}\, j_n \rightarrowtail H_n(C;G) \xrightarrow{h_n} \mathrm{im}\, h_n$$

is written as

$$\mathrm{coker}(i_n \otimes 1_G) \rightarrowtail H_n(C;G) \twoheadrightarrow \ker(i_{n-1} \otimes 1_G)$$

[4] The definition of splitting short exact sequence can be found in Exercise 1, Sect. II.3.

where the first and the third terms may be viewed in another way; we begin with $\operatorname{coker}(i_n \otimes 1_G)$.

(II.5.3) Lemma. $\operatorname{coker}(i_n \otimes 1_G) \cong H_n(C) \otimes G$.

Proof. Since by definition $\operatorname{coker}(i_n \otimes 1_G) = Z_n(C) \otimes G / \operatorname{im}(i_n \otimes 1_G)$, there exists a homomorphism

$$\phi\colon \operatorname{coker}(i_n \otimes 1_G) \to H_n(C) \otimes G$$

defined on the generators by $\phi[z \otimes g] := p(z) \otimes g$ (where $p\colon Z_n(K) \to H_n(C)$ is the natural projection). On the other hand for each $y \in H_n(C)$, we may choose an $x \in p^{-1}(y) \subseteq Z_n(C)$ and define

$$\psi\colon H_n(C) \otimes G \to \operatorname{coker}(i_n \otimes 1_G)$$

on the generators, with $\psi(y \otimes g) := [x \otimes g]$. The homomorphism ψ is well defined because, in view of the exactness of the long exact sequence, we have for each x' such that $p(x') = y$

$$x \otimes g - x' \otimes g = (x - x') \otimes g \in \ker j_n \cong \operatorname{im}(i_n \otimes 1_G),$$

that is to say, $[x \otimes g] = [x' \otimes g]$. The homomorphisms ϕ and ψ are clearly the inverse of each other and so $\operatorname{coker}(i_n \otimes 1_G) \cong H_n(C) \otimes G$. ■

The preceding lemma shows that $\operatorname{coker} i_n \otimes 1_G$ depends neither on $B_n(C) \otimes G$ nor on $Z_n(C) \otimes G$, but only on $H_n(C)$ (the cokernel of the monomorphism $i_n\colon B_n(C) \to Z_n(C)$) and on G. This fact suggests that the same may be true for $\ker(i_n \otimes 1_G)$ and indeed it is so.

(II.5.4) Theorem. *Let H be the cokernel of the the monomorphism $i\colon B \to Z$ between free Abelian groups and let G be any fixed Abelian group. Then, both the kernel and the cokernel of the homomorphism $i \otimes 1_G$ depend entirely on H and G.*

Moreover, $\operatorname{coker}(i \otimes 1_G) \cong H \otimes G$*, while* $\ker(i \otimes 1_G)$ *gives rise to a new covariant functor*

$$\operatorname{Tor}(-, G)\colon \mathbf{Ab} \longrightarrow \mathbf{Ab}$$

called torsion product.

Proof. Due to the fact that H is the cokernel of the monomorphism $i\colon B \to Z$, the bases of Z and B represent H with generators and relations; we then have a *free presentation* of H

$$B \overset{i}{\rightarrowtail} Z \overset{q}{\twoheadrightarrow} H.$$

Suppose that we had another free presentation of H

$$R \overset{j}{\rightarrowtail} F \overset{q'}{\twoheadrightarrow} H$$

and consider the following free chain complexes with augmentation to the Abelian group H (viewed as a $\mathbb{Z}$-module):[5]

1. (C,∂), with $C_1 = B$, $C_0 = Z$, $\partial_1 = i$, $\varepsilon = q$, $C_i = 0$ for all $i \neq 0,1$ and $\partial_i = 0$ for all $i \geq 2$;
2. (C',∂'), with $C'_1 = R$, $C'_0 = F$, $\partial'_1 = j$, $\varepsilon' = q'$, $C'_i = 0$ for all $i \neq 0,1$ and $\partial'_i = 0$ for all $i \geq 2$.

These chain complexes are free and acyclic; by Theorem (II.3.6), we obtain chain morphisms $f\colon C \to C'$ and $g\colon C' \to C$ whose composites fg and gf are chain homotopic to the respective identities. The tensor product with G is a functor that preserves compositions of morphisms; therefore, their tensor products by G produce the chain morphisms

$$f \otimes 1_G\colon C \otimes G \to C' \otimes G$$
$$g \otimes 1_G\colon C' \otimes G \to C \otimes G$$

and besides,

$$(f \otimes 1_G)(g \otimes 1_G) \text{ and } (g \otimes 1_G)(f \otimes 1_G)$$

are still chain homotopic to their respective identities. This means that the induced morphisms in homology

$$\ker(i \otimes 1_G) \underset{H_1(g \otimes 1_G)}{\overset{H_1(f \otimes 1_G)}{\rightleftarrows}} \ker(j \otimes 1_G)$$

are the inverse of each other and likewise for

$$\operatorname{coker}(i \otimes 1_G) \underset{H_0(g \otimes 1_G)}{\overset{H_0(f \otimes 1_G)}{\rightleftarrows}} \operatorname{coker}(j \otimes 1_G)$$

This implies that neither $\ker(i \otimes 1_G)$ nor $\operatorname{coker}(i \otimes 1_G)$ depends on the chosen presentation of H.

Hence, by following the argument in Lemma (II.5.3),

$$\operatorname{coker}(i \otimes 1_G) \cong H \otimes G$$

regardless of which free presentation of H we take.

[5] Chain complexes can be constructed over Λ-modules, with Λ a commutative ring with unit element.

We now focus our attention on functor $\mathrm{Tor}(-,G)$. For any $H \in \mathbf{Ab}$, we define the group $\mathrm{Tor}(H,G)$ as follows. Let $F(H)$ be the free group generated by all the elements of H; the function

$$q\colon F(H) \to H\ ,\ h \mapsto h$$

is an epimorphism of $F(H)$ onto H. Let $i\colon \ker q \to F(H)$ be the inclusion homomorphism; we then have a representation of H by free groups

$$\ker q \overset{i}{\rightarrowtail} F(H) \overset{q}{\twoheadrightarrow} H.$$

We define

$$\mathrm{Tor}(H,G) := \mathrm{coker}(i \otimes 1_G).$$

By the first part of the theorem, $\mathrm{Tor}(H,G)$ does not depend on the presentation of H. As for the morphisms, for any $\bar{f} \in \mathbf{Ab}(H,H')$, we choose the presentations $R \rightarrowtail F \twoheadrightarrow H$ and $R' \rightarrowtail F' \twoheadrightarrow H'$; by Theorem (II.3.6), we obtain a chain morphism f between the complexes C and C' (determined by $R \rightarrowtail F$ and $R' \rightarrowtail F'$, respectively) that extends $\bar{f}$ and is unique up to chain homotopy. By taking their tensor product by G and computing the homology groups, we obtain

$$H_1(f \otimes 1_G)\colon\ \mathrm{Tor}(H,G) \to \mathrm{Tor}(H',G)$$

which is, by definition, the result of applying the torsion product on $\bar{f}$. ■

When (C,∂) is the chain complex $(C(K),\partial)$ of an oriented simplicial complex K, the previous results prove the **Universal Coefficients Theorem in Homology**:

(II.5.5) Theorem. *The homology of a simplicial complex K with coefficients in an Abelian group G is determined by the following short exact sequences:*

$$H_n(K;\mathbb{Z}) \otimes G \rightarrowtail H_n(K;G) \twoheadrightarrow \mathrm{Tor}(H_{n-1}(K;\mathbb{Z}),G).$$

What is more, if G is free,

$$H_n(K;G) \cong H_n(K;\mathbb{Z}) \otimes G \oplus \mathrm{Tor}(H_{n-1}(K;\mathbb{Z}),G).$$

Let us now see what happens when $G = \mathbb{Q}$, the additive group of rational numbers. This group is not free, but it is *locally free*: we say that an Abelian group G is *locally free* if every finitely generated subgroup of G is free; in particular, due to the Finitely Generated Abelian Groups Decomposition Theorem (see p. 75), a finitely generated Abelian group is locally free if and only if it is torsion free. We now state the following

(II.5.6) Lemma. *If $i\colon A \to A'$ is a monomorphism and G is locally free, then*

$$i \otimes 1_G\colon A \otimes G \longrightarrow A' \otimes G$$

is a monomorphism.

In particular, the monomorphism

$$i_{n-1}\colon\ B_{n-1}(K) \longrightarrow Z_{n-1}(K)$$

determines the monomorphism

$$i_{n-1}\otimes 1_{\mathbb{Q}}\colon\ B_{n-1}(K)\otimes\mathbb{Q} \longrightarrow Z_{n-1}(K)\otimes\mathbb{Q}$$

and so

$$\mathrm{Tor}(H_{n-1}(K;\mathbb{Z}),\mathbb{Q}) = \ker(i_{n-1}\otimes 1_{\mathbb{Q}}) = 0\ .$$

Theorem (II.5.5) allows us to afirm that

$$H_n(K;\mathbb{Q}) \cong H_n(K;\mathbb{Z})\otimes\mathbb{Q}$$

and so, helped once more by the Finitely Generated Abelian Groups Decomposition Theorem, we say that $H_n(K;\mathbb{Q})$ is a rational vector space of dimension equal to the rank of $H_n(K;\mathbb{Z})$ (the nth- Betti number of K).

Exercises

1. Prove that, if A and B are free Abelian groups, then, $A\otimes B$ is a free Abelian group.

2. Let K be any simplicial complex. Prove that for every prime number p the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\ p\cdot -\ } \mathbb{Z} \xrightarrow{\ \mathrm{mod}\ p\ } \mathbb{Z}_p \longrightarrow 0$$

creates an exact sequence of homology groups

$$\cdots \longrightarrow H_n(K;\mathbb{Z}) \xrightarrow{\ p\cdot -\ } H_n(K;\mathbb{Z}) \xrightarrow{\ \mathrm{mod}\ p\ }$$

$$\xrightarrow{\ \mathrm{mod}\ p\ } H_n(K;\mathbb{Z}_p) \xrightarrow{\ \beta_p\ } H_{n-1}(K;\mathbb{Z}) \longrightarrow \cdots$$

called *Bockstein long exact sequence*. The homomorphism of Abelian groups

$$H_n(K;\mathbb{Z}_p) \xrightarrow{\ \beta_p\ } H_{n-1}(K;\mathbb{Z})$$

is called *Bockstein operator*.

3. (*Snake Lemma*) Consider the following commutative diagram, whose rows are exact sequences of Abelian groups:

$$\begin{array}{ccccccc} & & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow g & & \downarrow h & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & & \end{array}$$

Prove that there exists a homomorphism d that turns the sequence

$$\ker f \longrightarrow \ker g \longrightarrow \ker h \xrightarrow{d} \operatorname{coker} f \longrightarrow \operatorname{coker} g \longrightarrow \operatorname{coker} h$$

into an exact sequence; also, by using this Lemma, give an alternative proof to Theorem (II.3.1).

4. (*General form of the Five Lemma*) Let

$$\begin{array}{ccccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & D & \xrightarrow{k} & E \\ \alpha\downarrow & & \beta\downarrow & & \gamma\downarrow & & \delta\downarrow & & \varepsilon\downarrow \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & D' & \xrightarrow{k'} & E' \end{array}$$

be a commutative diagram of Abelian groups with exact rows. Prove that:

- If α is surjective and β, δ are injective, then γ is injective;
- If ε is injective and β, δ are surjective, then γ is surjective.

It follows directly from these results that, if α, β, δ, and ε are isomorphisms, then also γ is an isomorphism.

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