A DOUBLE BICATEGORY OF COBORDISMS WITH CORNERS

JEFFREY MORTON

ABSTRACT. Interest in cobordism categories arises for various reasons, in areas from topology to theoretical physics. These categories have manifolds as objects, and as morphisms, have cobordisms between them - that is, manifolds of one dimension higher whose boundary decomposes into the source and target. The principle that the boundary of a boundary is the empty set means that this formulation cannot account for cobordisms between manifolds with boundary, as required, for instance, to describe evolution of *open strings* in string theory. We describe a category-theoretic framework in which this can be expressed, in the form of a *Verity double bicategory.* This is similar to a double category (a category object in **Cat**), but with properties holding only up to certain 2-morphisms.. We sketch how this is a special case of a more general "n-tuple bicategory". Then we show how a broad general class of examples arise from a construction involving spans (or cospans) in any chosen category, and how this gives cobordisms between cobordisms when we start with a category of suitable smooth spaces.

1. INTRODUCTION

The purpose of this paper is to describe a double bicategory of cobordisms with corners.

A cobordism between manifolds S and S' is a manifold with boundary M such that ∂M is the disjoint union of S and S', which we think of as an arrow $M : S \to S'$. One can define composition of cobordisms, by gluing along components of the boundary, leading to the definition of a category **nCob** of n-dimensional cobordisms between $(n-1)$-dimensional manifolds. It is natural to consider the possibility that S and S' themselves have boundary, and ask if one can similarly describe cobordisms between them. In particular, we are interested in the case where $S : X \to Y$ and $S' : X' \to Y'$ are already themselves cobordisms. Such cobordisms are always manifolds with corners. Here we shall describe a formalism for describing the ways such cobordisms can be glued together. Louis Crane has written a number of papers on this issue, including one with David Yetter [CY] which describes a *bicategory* of such cobordisms. Here, we construct a *double bicategory*, $\mathbf{nCob}_2$.

One motivation for doing this comes from the fact that interest in **nCob** has been encouraged by Michael Atiyah's axiomatic description of topological quantum field theories, or *TQFTs* ([Ati1, Ati2]). A TQFT assigns a space of states to each manifold, and a linear transformation between states to cobordisms. Ruth Lawrence [Law] described the notion of an extended TQFT. These are theories similar to TQFT's, for which the theory is defined not on cobordisms, but on manifolds with corners. Crane and Yetter [CY], describe the algebraic structure of TQFT's and extended TQFT's. Baez and Dolan [BaDo] summarize the connection between TQFT's and higher category theory, in the form of the *Extended TQFT Hypothesis*, suggesting that all extended TQFT's can be viewed as representations of a certain kind of "free n-category".

The kind of n-category we are interested in here is a common generalization of a double category and a bicategory. Double categories, introduced by Ehresmann ([Eh1, Eh2]), may be seen as an "internal" category in **Cat** - that is, a structure with a category of objects and a category of morphisms. Less abstractly, it has objects, horizontal and vertical morphisms (edges in an underlying 2-graph), and squares (cells in the underlying 2-graph):

(1)
$$\begin{array}{ccc} x & \xrightarrow{\phi} & x' \\ \Big\downarrow f & \Swarrow F & \Big\downarrow f' \\ y & \xrightarrow{\hat{\phi}} & y' \end{array}$$

These can be composed in geometrically obvious ways to give diagrams analogous to those in ordinary category theory. Moskaliuk and Vlassov [MV] discuss the application of double categories to mathematical physics, particularly TQFT's, and dynamical systems with changing boundary conditions - that is, with inputs and outputs.

Double categories are too strict to be really natural for our purpose, however. In particular, describing cobordisms as categories or double categories requires us to take diffeomorphism classes of cobordisms, not cobordisms themselves, as morphisms. So we will consider a weakening of this structure, in the sense that axioms for a double category giving equations (such as associativity) will be true only up to a certain 2-morphism. This is analogous to the way in which a bicategory is a weakening of the idea of a category. These are like categories with an extra level of *2-morphism* between morphisms, and such that equations in the axioms for a category are replaced by 2-isomorphisms, which look like:

(2)
$$x \underset{g}{\overset{f}{\rightrightarrows}} y, \qquad \alpha : f \Rightarrow g$$

Bicategories, however, are not really what we want to describe $\mathbf{nCob}_2$, either, since we want to describe systems with changing boundary conditions, and the most natural way to do this is by allowing both initial and final states, and these changing conditions, as part of the boundary. On the other hand, we show in theorem 1 that double bicategories satisfying certain conditions are equivalent to bicategories - and in fact $\mathbf{nCob}_2$ is an example of this.

In figure 1 we see a manifold with corners which illustrates these points and provides some motivating intuition. This can be seen a cobordism from the pair of annuli at the top to the two-punctured disc at the bottom. These in turn can be thought of as cobordisms - respectively - from one pair of circles to another, and from one circle to two circles. The large cobordism has other boundary components: the outside boundary is itself a cobordism from two circles to one circle; the inside boundary (in dotted lines) is a cobordism from one pair of circles to another pair. We could "compose" this with another such cobordism with corners by gluing along any of the four boundary components: top or bottom, inside or outside.

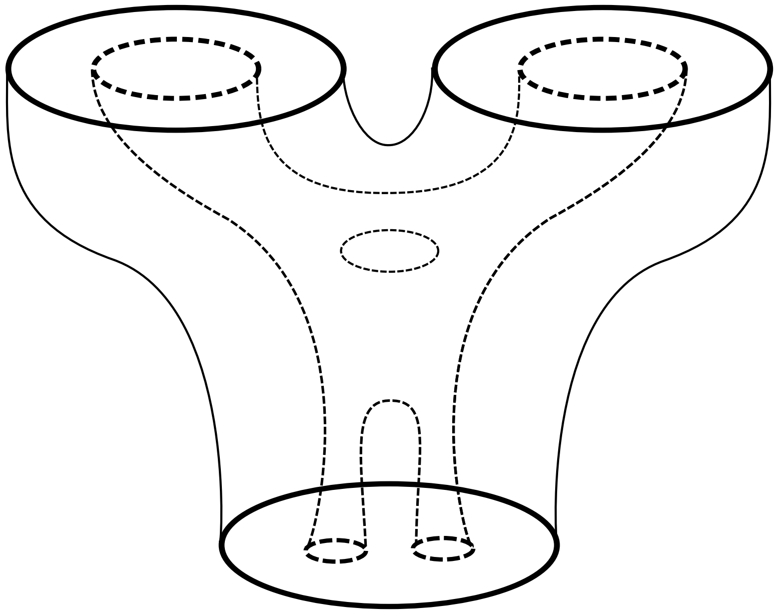


FIGURE 1. A Cobordism With Corners

This involves attaching another such cobordism having a boundary component diffeomorphic to any of these.

The structure we use is a *Verity double bicategory*. There is some ambiguity here since the concept this appears to describe is an internal bicategory in **Bicat** (the category of all bicategories). This is analogous to the definition of double category. Indeed, it is what we will mean by a double bicategory, and we discuss these in section 6. However, the term double bicategory seems to have been originally introduced by Dominic Verity [Ver], and uses it to refer to a somewhat different structure - which is, in fact, the one we want to use. We call these Verity double bicategories. In section 3 we describe some of the necessary mathematical background of bicategories and double categories, and briefly describe standard examples of these from homotopy theory, which provide some topological motivation for these categorical concepts.

In section 4 we describe double bicategories in the sense of Verity (which we call *Verity double bicategories* to distinguish them from internal bicategories in **Bicat**). These go further than many efforts to weaken the concept of a double category, such as the "weak double categories" discussed by Marco Grandis and Robert Paré ([GP1], [GP2]), or the "pseudo-categories" discussed by Martins-Ferreira [Mar]. Thomas Fiore in [Fio] describes how these arise by "categorification" of the theory of categories, and describes examples motivated by conformal field theory. In these examples, the composition in double categories are weakened in only one direction. That is, the associativity of composition, and unit laws, apply only up to certain higher morphisms, called *associators* and *unitors* - but only in the horizontal direction (equivalently, only in the vertical direction). In double bicategories, this is true in both directions. In section 5.3 we prove the main theorem of the paper, that such $\mathbf{nCob_2}$ indeed forms a Verity double bicategory. This uses a technical lemma involving double bicategories. In section 4.3 we prove that a Verity double bicategory gives a bicategory, just as any double category corresponds to a strict 2-category.

To finish section 4, we describe a general class of examples of double bicategories, analogous to the result that Span($\mathbf{C}$) is a bicategory. This was shown by Jean Bénabou in [Ben], who introduced both the concept of a span, and of bicategories. The "double spans" we describe here give a broad class of examples, and in particular, we can use them to derive the fact that there is a double bicategory of cobordisms with corners.

We describe the background for such cobordisms in section 5. Gerd Laures [Laur] discusses the general theory of cobordisms of manifolds with corners. In the terminology used there, introduced by Jänich [Jan], what we primarily discuss in this paper are $\langle 2 \rangle$-manifolds: in particular, the *codimension* of the manifold is 2. That is, the manifold M (whose dimension is $\dim(M) = n$) will have a boundary ∂M, which will in turn be composed of faces which are manifolds with boundary, of dimension $(n-1)$. However, the boundaries of these faces will be closed manifolds: they are manifolds of dimension $(n-2)$. This separates into *faces*. For us, the faces decompose into components, which are the source and the target in both horizontal and vertical directions. The corners, faces of codimension 2, are the source and target of these. We call the Verity double bicategory obtained this way $\mathbf{nCob}_2$.

Having done this, we continue, in section 7 by briefly examining a low-dimensional example, studied in a "decategorified" setting, without using Verity double bicategories in [LP]. This is the case of "open-closed strings", on which Lauda and Pfeiffer describe a certain kind of TQFT. We see how their category of "open-closed strings" is related to the Verity double bicategory $\mathbf{2Cob_2}$.

In section 8, we suggest some further directions to expand on this work. We consider the program of describing $\mathbf{nCob}_2$ in terms of generators and relations. Also, we discuss the problem of extending the idea of double bicategories to cobordisms of higher codimension than 2. We suggest a way in which this could be approached.

Finally, we should note here that there are at least two audiences for this paper. Cobordisms are of interest to topologists and, through TQFT's to those interested in mathematical physics. On the other hand, the notion of a double bicategory may appeal to those interested in category theory, and n-categories in particular. The aim here is to provide something of interest to each. Readers mainly interested in category theory, may find section 6 of most interest, while topologically-inclined readers may consider it merely a repository of two technical lemmas and skip it. Likewise, most discussion of cobordisms with corners is in section 5, and in particular section 5.1 describes a "collaring" condition essential to make sure that composition of cobordisms gives a smooth cobordism. This is important from a topological point of view and has an important implication for what we take our cobordisms to be. Categorically inclined readers may find this less interesting since cobordisms are, from that point of view, only a special case of a very general class of examples of Verity double bicategories. Readers should feel free to skip to the sections of most interest, which should be relatively self-contained.

2. The Category **nCob**

In this section, we review the structure of the symmetric monoidal category **2Cob** which we generalize in this paper. Cobordism theory goes back to the work of René Thom [Tho], and is closely related to homotopy theory. A good introductory discussion suitable for our needs is found, e.g. in Hirsch [Hir]. There is substantial research on many questions in cobordism theory, such as *references.* Two manifolds S_1, S_2 are *cobordant* if there is a compact manifold with boundary, M, such that ∂M is isomorphic to the disjoint union of S_1 and S_2. This M is called a *cobordism* between S_1 and S_2. We note that there is some similarity between this concept and that of homotopy of paths, except that such homotopies are understood as embedded in an ambient space. We will return to this in section 3.5. Our aim here is to describe a generalization of categories of cobordisms. To begin with, we recall some of the structure of **nCob**, and particularly **2Cob**, to recall why this is of interest.

2Cob is the category whose objects are one-dimensional compact oriented manifolds, and whose morphisms are diffeomorphism classes of two-dimensional compact oriented cobordisms between such objects. That is, the objects are collections of circles, and the morphisms are (diffeomorphism classes of) manifolds with boundary, whose boundaries are broken into two parts, which we consider their source and target. We think of the cobordism as "joining" two manifolds, rather as a relation joins two sets, in the category of sets and relations. More generally, **nCob** is the category whose objects are (compact, oriented) $(n-1)$-dimensional manifolds, and whose morphisms are diffeomorphism classes of compact oriented n-dimensional cobordisms.

2.1. **Presentation.** It was shown by Abrams [Abr] that **2Cob** can be seen as the free symmetric monoidal category on a Frobenius object. (Another good exposition of this was developed by Joachim Kock [Ko].) This amounts to saying that **2Cob** is generated from five generators, called the **unit**, **counit**, **multiplication**, **comultiplication**. They include cobordisms: taking the empty set to the circle (the unit); taking two circles to one circle (the multiplication); adjoints of each of these (counit and comultiplication repsectively). The "Frobenius object" appears here as the circle, equipped with these morphisms, which are illustrated in figure 2.

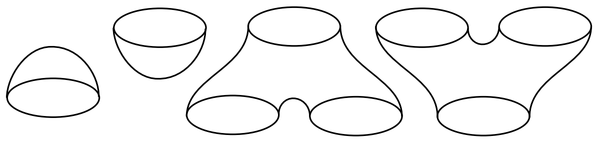


Figure 2. Generators of **2Cob**

The category **2Cob** also includes **identity** cobordism, taking the circle to itself by $S^1 \times I$; and the **switch** cobordism, exchanging the order of two circles by two cylinders (this gives the symmetry for the monoidal operation). These are required to exist by the assumption that **2Cob** is a free symmetric monoidal category. They are illustrated in figure 3.

Two proofs can be given for the fact than **2Cob** is generated by these cobordisms, and each relies on some special conditions satisfied by 2D cobordisms. The first

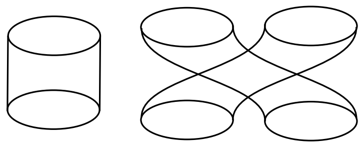


FIGURE 3. Morphisms Required for **2Cob** to be a Symmetric Monoidal Category

is that 2-dimensional manifolds with boundary can be completely classified up to diffeomorphism class by genus and number of punctures. The second is that we can use the results of Morse theory to decompose any such surface with a smooth Morse function into $[0,1]$ into a composite (in the sense of composition of morphisms in **2Cob**) of pieces. In each piece, there is just one "topology change" (a value in $[0,1]$ where the preimage changes topology). We will return to this point when we discuss the question of how to present $\mathbf{nCob_2}$ in terms of generators.

So far, we have described the generators for the category **2Cob**, but not yet how the composition operation for morphisms works. The main idea is that we compose cobordisms by identifying their boundaries - however, since the morphisms in **2Cob** are *diffeomorphism* classes of manifolds with boundary, some extra considerations are needed to ensure that the composite is equipped with a differentiable structure.

In particular, the *collaring theorem* means that any manifold with boundary, M can be equipped with a "collar": an injection $\phi : \partial M \times [0,1] \to M$ such that $\phi(x,0) = x, \forall x \in \partial M$. The idea is that, while we can compose *topological* cobordisms along their boundaries, we should compose *smooth* cobordisms M_1 and M_2 along collars. This ensures that every point - including points on the boundary of M_i - will have a neighborhood with a smooth coordinate chart. Section 5.1 describes this in detail for a more general setting.

Moreover, as a *monoidal* category, **2Cob** must have a tensor product operation. For objects, this is just the disjoint union: given objects $\mathbf{m}, \mathbf{n} \in \mathbf{2Cob}$, consisting of collections of m and n circles respectively, the object $\mathbf{m} \otimes \mathbf{n}$ is the disjoint union of $\mathbf{m}$ and $\mathbf{n}$ - a collection of $m+n$ circles. The tensor product of two cobordisms $\mathbf{C_1} : \mathbf{m_1} \to \mathbf{n_1}$ and $\mathbf{C_2} : \mathbf{m_2} \to \mathbf{n_2}$ is likewise the disjoint union of the two cobordisms, giving $\mathbf{C_1} \otimes \mathbf{C_2} : \mathbf{m_1} \otimes \mathbf{m_2} \to \mathbf{n_1} \otimes \mathbf{n_2}$.

2.2. **Applications.** The category **2Cob** is particularly interesting in the study of topological quantum field theories (TQFT's), as formalized by Michael Atiyah ([Ati1, Ati2]). Each TQFT is a functor $F : \mathbf{2Cob} \to \mathbf{Vect}$. The presentation of **2Cob** in terms of its generators means that this immediately defines an algebraic structure with a unit, counit, multiplication, comultiplication, and identity (a bialgebra). The fact that **2Cob** is a *symmetric* monoidal category means that this structure satisfies the axioms of a Frobenius algebra.

One may wish to describe an "extended topological quantum field theory" in the same format. These are topological field theories which are defined not just on manifolds with boundary, but also on manifolds with corners. This idea is described by Ruth Lawrence in [Law]. In particular, what we are interested in here is that, instead of using a category of cobordisms between manifolds, we would want to use some structure of cobordisms between *cobordisms* between manifolds, which we tentatively call $\mathbf{nCob_2}$. However, to do this, we must use a structure with more elaborate than a mere category.

So next we describe such a structure, a *Verity double bicategory*, and show how the putative $\mathbf{nCob_2}$ is an example, and indeed a special case of a broad class of examples.

3. Bicategories and Double Categories

We want to give a description of a *Verity double bicategory.* Weakening a concept X in category theory generally involves creating a new concept in which equations in the original concept are replaced by isomorphisms. Thus, we say that the old equations hold only "up to" isomorphism in the weak version of X, and say that when they hold with equality, we have a "strict X". Thus, before describing our newly weakened concept, it makes sense to recall how this process works, and examine the strict form of the concept we want to weaken. So we begin by reviewing bicategories and double categories.

3.1. **2-Categories.** A category $\mathbf{E}$ is **enriched over** a category $\mathbf{C}$ (which must have products) when for $x, y \in \mathbf{E}$ we have $\hom(x, y) \in \mathbf{C}$. A special case of this occurs in "closed" categories, which are enriched over themselves - examples include **Set** (since there is a set of maps between any two sets) and **Vect** (since the linear operators between two vector spaces form a vector space).

A **2-category** is a category enriched over **Cat**. That is, if $\mathbf{C_2}$ is a 2-category, and $x, y \in \mathbf{C_2}$), then $\hom(x, y) \in \mathbf{Cat}$. Thus, there are sets of objects and morphisms in $\hom(x, y)$ itself, with the usual category axioms. We describe a 2-category as having **objects**, **morphisms** between objects, and **2-morphisms** between morphisms. The morphisms of $\mathbf{C_2}$ are the objects of the hom-categories, and the 2-morphisms of $\mathbf{C_2}$ are the morphisms of the hom-categories. We depict these as in diagram (2). These have a composition operation between morphisms, and also a "horizontal" composition, which we denote $\circ$, and a "vertical" composition, denoted $\cdot$, between 2-morphisms.

Furthermore, for all $x, y, z \in \mathbf{C_2}$, the composition operation

$$\circ : \hom(x, y) \times \hom(y, z) \to \hom(x, z) \tag{3}$$

must be a functor between hom-categories. This requirement means that the **interchange law** holds:

$$(\alpha \circ \beta) \cdot (\alpha' \circ \beta') = (\alpha \cdot \alpha') \circ (\beta \cdot \beta') \tag{4}$$

Now, in a 2-category, the associative law holds strictly: that is, for morphisms $f \in \hom(w, x)$, $g \in \hom(x, y)$, and $h \in \hom(y, z)$, we have the two possible triple-compositions in $\hom(w, z)$ the same, namely $f \circ (g \circ h) = (f \circ g) \circ h$. This is one of the axioms for a category - that is, a category enriched over **Set**. Since a 2-category is enriched over **Cat**, however, a weaker version of this rule is possible, since $\hom(w, z)$ is no longer a set in which elements can only be equal or unequal: it is a category, where it is possible to speak of isomorphic objects. This fact leads to the notion of bicategories.

3.2. **Bicategories.** Once we have the concept of a 2-category, we can *weaken* this concept, giving the idea of a **bicategory**. The definition is similar to that for a 2-category, but we only insist that the usual equations should be natural isomorphisms (satisfying some equations). That is, the following diagrams should commute up to natural isomorphisms:

$$\begin{array}{ccc} \hom(w,x)\times\hom(x,y)\times\hom(y,z) & \xrightarrow{1\times\circ} & \hom(w,x)\times\hom(x,z) \\ \Big\downarrow{\scriptstyle \circ\times 1} & & \Big\downarrow{\scriptstyle \circ} \\ \hom(w,y)\times\hom(y,z) & \xrightarrow[\circ]{} & \hom(w,z) \end{array} \tag{5}$$

and

$$\begin{array}{ccc} \hom(x,y)\times\mathbf{1} & & \\ \Big\downarrow{\scriptstyle \mathrm{id}\times !} & \searrow^{\pi_1} & \\ \hom(x,y)\times\hom(x,x) & \xrightarrow{\circ} & \hom(x,y) \end{array} \tag{6}$$

and

$$\begin{array}{ccc} \mathbf{1}\times\hom(x,y) & & \\ \Big\downarrow{\scriptstyle !\times\mathrm{id}} & \searrow^{\pi_2} & \\ \hom(y,y)\times\hom(x,y) & \xrightarrow{\circ} & \hom(x,y) \end{array} \tag{7}$$

That is: given $(f,g,h) \in \hom(w,x)\times\hom(x,y)\times\hom(y,z)$, there should be an isomorphism $a_{f,g,h} \in \hom(w,z)$ with $a_{f,g,h} : (f\circ g)\circ h \to f\circ(g\circ h)$; and isomorphisms $r_f : f\circ 1_x$, $l_f : 1_y \circ f$. The equations these satisfy are *coherence laws*. MacLane's Coherence Theorem shows that all such equations follow from two generating equations: the pentagon identity, and the unitor law:

In a category, the associativity property stated that two composition operations can be performed in either order and the results should be equal - equality is the only sensible relation between a pair of morphisms in a category. There is an analogous statement for the associator 2-morphism: two different ways of composing it should yield equal results (since equality is the only sensible relation between a pair of 2-morphisms in a bicategory). This property is the pentagon identity:

$$\begin{array}{ccccc} & & (f\circ g)\circ(h\circ j) & & \\ & \nearrow^{a_{f\circ g,h,j}} & & \searrow^{a_{f,g,h\circ j}} & \\ ((f\circ g)\circ h)\circ j & & & & f\circ(g\circ(h\circ j)) \\ \Big\downarrow{\scriptstyle a_{f,g,h}\circ 1_j} & & & & \Big\uparrow{\scriptstyle 1_f\circ a_{g,h,j}} \\ (f\circ(g\circ h))\circ j & & \xrightarrow[a_{f,g\circ h,j}]{} & & f\circ((g\circ h)\circ j) \end{array} \tag{8}$$

Similarly, the unit laws satisfy the property that the following commutes:

$$\begin{array}{ccc} (g\circ 1_y)\circ f & \xrightarrow{a_{g,1_y,f}} & g\circ(1\circ f) \\ \Big\downarrow{\scriptstyle r_g\times 1_f} & \swarrow_{1_g\times l_f} & \\ g\circ f & & \end{array} \tag{9}$$

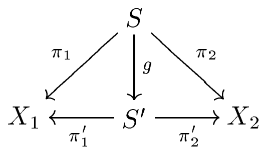
This last change is the sort of weakening we want to apply to the concept of a double category. Following the same pattern, we will first describe (in section 3.4) the strict notion before describing how to weaken it in section 4. First, however, we will describe a standard, quite general, example of bicategory, which we will generalize to give examples of double bicategories in section 4.4.

3.3. **Bicategories of Spans.** Jean Bénabou [Ben] introduced bicategories in a 1967 paper, and one broad class of examples introduced there comes from the notion of a *span*.

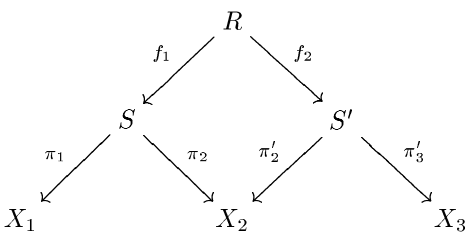
Definition 1. *Given any category* $\mathbf{C}$*, a* ***span*** (S, π_1, π_2) *between objects* $X_1, X_2 \in \mathbf{C}$ *is a diagram in* $\mathbf{C}$ *of the form*

(10) $$P_1 \xleftarrow{\pi_1} S \xrightarrow{\pi_2} P_2$$

Given two spans (S, s, t) *and* (S', s', t') *between* X_1 *and* X_2 *between a* ***morphism of spans*** *is a morphism* $g : S \to S'$ *making the following diagram commute:*

(11)


Definition 2. *Composition of spans* S *from* X_1 *to* X_2 *and* S' *from* X_2 *to* X_3 *is given by a pullback: that is, an object* R *with maps* f_1 *and* f_2 *making the following diagram commute:*

(12)


which is terminal among all such objects. That is, given any other Q *with maps* g_1 *and* g_2 *which make the analogous diagram commute, these maps factor through a unique map* $Q \to R$*.* R *becomes a span from* X_1 *to* X_3 *with the maps* $\pi_1 \circ f_1$ *and* $\pi_2 \circ f_2$*.*

The span construction has a dual concept:

Definition 3. *A* ***cospan*** *in* $\mathbf{C}$ *is a span in* $\mathbf{C}^{\mathrm{op}}$*, morphisms of cospans are morphisms of spans in* $\mathbf{C}^{\mathrm{op}}$*, and composition of cospans is given by pullback in* $\mathbf{C}^{\mathrm{op}}$*. That is, by a pushout in* $\mathbf{C}$*.*

One fact about (co)spans which is important for our purposes is that any category $\mathbf{C}$ with limits (colimits, respectively) gives rise to a bicategory of spans (or cospans). This relies in part on the fact that the pullback is a universal construction (universal properties of Span($\mathbf{C}$) are discussed by Dawson, Paré and Pronk [DPP]).

Remark 1. [Ben]**, ex. 2.6** Given any category $\mathbf{C}$ with all limits, there is a bicategory Span($\mathbf{C}$), whose objects are the objects of $\mathbf{C}$, whose *hom*-sets of morphisms

Span($\mathbf{C}$)(X_1, X_2) consist of all spans (cospans) between X_1 and X_2, with composition as defined above, and whose bigons are morphisms of spans (cospans). Span($\mathbf{C}$) (or Cosp($\mathbf{C}$)) as defined above forms a bicategory.

This is a standard result, first shown by Jean Benabou in [Ben], as one of the first examples of a bicategory. We briefly describe the proof:

The identity for X is $X \overset{id}{\leftarrow} X \overset{id}{\rightarrow} X$, which is easy to check.

The associator arises from the fact that the pullback is a *universal* construction. Given morphisms in Span($\mathbf{C}$) $f : X \to Y$, $g : Y \to Z$, $h : Z \to W$, the composites $((f \circ g) \circ h)$ and $(f \circ (g \circ h))$ are pullbacks consisting of objects O_1 and O_2 with maps into X and W. The universal property of pullbacks gives an isomorphism between O_1 and O_2. These isomorphisms satisfy the pentagon identity since they are unique (in particular, both sides of the pentagon give the same isomorphism).

It is easy to check that hom(X_1, X_2) is a category, since it inherits all the usual properties from $\mathbf{C}$.

3.4. **Double Categories.** A (strict) double category can be thought of as an internal category in **Cat**. That is, it is a model of the theory of categories, denoted **Th(Cat)**, in **Cat**. This **Th(Cat)** consists of a category with two objects, Obj and Mor with morphisms of the form:

$$\mathrm{Mor} \underset{t}{\overset{s}{\rightrightarrows}} \mathrm{Obj} \tag{13}$$

subject to some axioms. In particular, the composition operation is a partially defined operation on pairs of morphisms. In particular, there is a collection of composable pairs of morphisms, namely the fibre product **Pairs** = Mor $\times_{\mathbf{Obj}}$ Mor, which is a pullback of the two arrows from Mor to **Obj**. That is, **Pairs** is an equalizer in the following diagram:

$$\begin{array}{ccccc} & & & \mathrm{Mor} & \\ & & {\scriptstyle \pi_1}\nearrow & & \searrow{\scriptstyle t} \\ \mathbf{Pairs} & \xrightarrow{i} & \mathrm{Mor}^2 & & \mathrm{Obj} \\ & & {\scriptstyle \pi_2}\searrow & & \nearrow{\scriptstyle s} \\ & & & \mathrm{Mor} & \end{array} \tag{14}$$

(Note that we assume the existence of pullbacks, here - in fact, **Th(Cat)** is a *finite limit theory.*) The composition map $\circ$: **Pairs** $\to$ Mor satisfies the usual properties for composition.

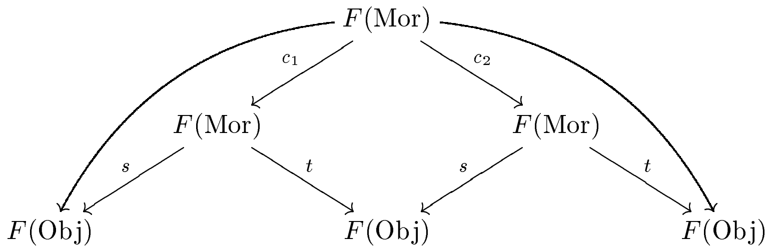
There is also an identity for each object: there is a map Obj $\overset{1}{\to}$ Mor, such that for any morphism $f \in$ Mor, we have $1_{s(f)}$ and $1_{t(f)}$ are composable with f, and the composite is f itself.

A model of **Th(Cat)** in **Cat** is a (limit-preserving) functor

$$F : \mathbf{Th(Cat)} \to \mathbf{Cat}$$

This gives a structure having a category **Ob** of objects and a category **Mor** of morphisms, with two functors s ("source") and t ("target") satisfying the usual category axioms. In particular, we can describe composition as a pullback construction in this category, which makes sense since the functor preserves finite limits (including pullbacks):

(15)



A category is a model of the theory **Th(Cat)** in **Set**, and we understand this to mean that when two morphisms f and g have the target of f the same as the source of g, there is a composite morphism from the source of f to the target of g. In the case of a double category, we have a model of **Th(Cat)** in **Cat**, so that $F(\text{Obj})$ and $F(\text{Mor})$ are categories and $F(s)$ and $F(t)$ are functors, we have the same condition for both objects and morphisms - subject to the compatibility conditions for these two maps which any functor must satisfy.

We thus have sets of objects and morphisms in **Ob**, which of course must satisfy the usual axioms. The same is true for **Mor**. The category axioms for the double category are imposed on top of these properties, with compatibility conditions between the two. The result is that we can think of both the objects in **Mor** and the morphisms in **Ob** as acting like morphisms between the objects in **Ob**, in a way compatible with the source and target maps. A double category can be thought of as including within it the morphisms of two potentially different categories on the same collection of objects. These are customarily called the *horizontal* and *vertical* morphisms, intuitively capturing the picture:

(16)
$$\begin{array}{ccc} x & \xrightarrow{\phi} & x' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ y & \xrightarrow[\hat{\phi}]{} & y' \end{array}$$

Here, the objects in the diagram can be thought of as objects in $F(\text{Obj})$, the vertical morphisms f and f' can be thought of as morphisms in $F(\text{Obj})$ and the horizontal morphisms ϕ and $\hat{\phi}$ as objects in $F(\text{Mor})$. In fact, there is enough symmetry in the axioms for an internal category in **Cat** that we can adopt either convention. However, we also have morphisms in Mor. We represent these as two-cells, or *squares*, like the square S represented in this diagram:

(17)
$$\begin{array}{ccc} x & \xrightarrow{\phi} & x' \\ {\scriptstyle f}\downarrow & \swarrow_S & \downarrow{\scriptstyle f'} \\ y & \xrightarrow[\hat{\phi}]{} & y' \end{array}$$

The fact that the composition map $\circ$ is a functor means that horizontal and vertical composition of squares commutes.

3.5. **Topological Examples.** We can illustrate bicategories and double categories in an elementary topological setting, namely from homotopy theory. This was the source of much of the original motivation for higher-dimensional category theory. Moreover, as we have already remarked in 2, there are close connections between cobordism and homotopy. These examples will turn out to suggest how to describe Verity double bicategories of cobordisms.

Our first example is perhaps the original motivating example of a bicategory.

Example 1. Given a space S in the category **Top** of topological spaces, we might wish to define a category $\mathrm{Path}(S)$ whose objects are points of X, and whose morphisms are paths in S. That is, a morphism in $\mathrm{Path}(S)$ from a to b is a map $\gamma : [0,1] \to X$ such that $\gamma(0) = a$ and $\gamma(1) = b$. The obvious composition rule for $\gamma_1 \in \hom(a,b)$ and $\gamma_2 \in \hom(b,c)$ is that

$$\gamma_1;\gamma_2(x) = \begin{cases} \gamma_1(2x) & \text{if } x \in [0, \frac{1}{2}) \\ \gamma_2(2x-1) & \text{if } x \in [\frac{1}{2}, 1] \end{cases} \tag{18}$$

However, this composition rule is not associative, and resolving this involves, either implicitly or explicitly, use of a bicategory. We get this bicategory $\mathrm{Path}_2(S)$, by first defining, for $a, b \in S$, a category $\hom(a,b)$ with:

- objects: paths from a to b
- morphisms: homotopies between paths, namely a homotopy from γ_1 to γ_2 is $H : [0,1] \times [0,1] \to S$ such that $H(x,0) = \gamma_1(x)$, $H(x,1) = \gamma(x)$, $H(0,y) = a$, $H(1,y) = b$ for all $(x,y) \in [0,1] \times [0,1]$.

Then we have a unit law for the identity morphism (the constant path) at each point, and an associator for composition. Both of these are homotopies which reparametrize composite paths.

Finally, we note that, if we define horizontal and vertical composition of homotopies in the same way as above (in each component), then this composition is again not associative. So to get around this, we say that the bicategory we want has its hom-categories $\underline{\hom(a,b)}$, where the morphisms are *isomorphism classes* of homotopies. The isomorphisms in question will not be homotopies themselves (to avoid an infinite regress), but rather smooth maps which fix the boundary of the homotopy square.

We call the resulting bicategory $\mathrm{Path}_2(S)$.

A similar construction is possible for a double category.

Example 2. A double category is a model of **Th(Cat)** in **Cat**, and we have seen that it is analogous to a bicategory. So we would like to construct one analogous to the bicategory in Example 1, we construct a model having the following:

- A category Obj of objects: we take this to be $\mathrm{Path}(S)$, the path category of S.
- A category Mor of morphisms: we take this to have the following data:
  - objects: paths $\gamma : [m,n]$ in S
  - morphisms: homotopies $H : [p,q] \times [m,n]$ between paths (these have source and target maps which are just $s : H(-,-) \to H(-,m)$ and $t : H(-,-) \to H(-,n)$.

  These categories have source and target maps s and t which are functors from Mor to Obj. The object map for s is just evaluation at 0, and for t it is evaluation at 1. The morphism maps for these functors are $s : H(-,-) \to H(p,-)$ and $t : H(-,-) \to H(q,-)$.

We call the result the double category of homotopies, $\mathbf{H}(S)$.

We observe here that the double category $\mathbf{H}(S)$ is similar to the bicategory $\mathrm{Path}_2(S)$ in one sense. Both give a picture in which objects are points in a topological space, morphisms are 1-dimensional objects (paths), and higher morphisms involve 2-dimensional objects (homotopies). There are some obvious differences, however. The most obvious is that $\mathrm{Path}_2(S)$ involves only homotopies with fixed endpoints: its 2D objects are *bigons*, whereas in $\mathbf{H}(S)$ the 2D objects are "squares" (or images of rectangles under smooth maps).

A more subtle difference, however, is that, in order to make composition strictly associative in $\mathbf{H}(S)$, it was necessary to change how we parametrize the homotopies. There are no associators here, and so we make sure composition is strict by not rescaling our source object (the product of two intervals) as we did in $\mathrm{Path}_2(S)$.

This is rather unsatisfactory, and in fact improving it leads to a general definition of a *double bicategory*, which has a large class of examples - namely, *double spans*, including as a special, restricted case, the double bicategory of cobordisms with corners we want.

4. Double Bicategories

4.1. Weak Double Categories, Double Bicategories, and Internal Bicategories. We wish to describe a structure which is sufficient to capture the possible compositions of cobordisms with corners just as **2Cob** does for cobordisms. These not only have composition along the manifolds with boundary which form their source and target, but also along the boundaries of those manifolds (and along the boundaries of the cobordisms, which join these). However, to allow the boundaries to vary, we do not want to consider them as diffeomorphism classes of cobordisms, but simply as cobordisms. However, composition is then not strictly associative, but only up to diffeomorphism.

Thus, we want something like a double category, but we must weaken the axioms for a double category, just as bicategories were defined by weakening those for a category. The concept of a "weak double category" has been defined (for instance, see [GP1] and [Fio], where these are seen as "Pseudo Double Categories"), but the weakening only occurs in only one direction - either horizontal or vertical. In the other direction, the category axioms hold strictly. In a sense, this is because the weakening uses the squares of the double category as 2-morphisms - in particular, squares with two sides equal to the identity. Trying to do this in both directions leads to difficultly.

In particular, if we have associators for horizontal morphisms given by squares of the form:

$$\begin{array}{ccccc} a & \xrightarrow{f;g} & c & \xrightarrow{h} & d \\ \downarrow & & \Downarrow \alpha_{f,g,h} & & \downarrow \\ a & \xrightarrow{f} & b & \xrightarrow{g;h} & d \end{array} \tag{19}$$

then unless composition of vertical morphisms is strict, then to make a equation (for instance, the pentagon equation) involving this square, we would need to use unit laws (or associators) in the vertical direction to perform this composition. This would again be a square with identities on two sides, and the problem arises again. In fact, there is no consistent way to do this. Instead, we need to introduce a new kind of 2-morphism separate from the squares, as we shall see in section 4.2. The result is what Dominic Verity has termed a double bicategory [Ver].

The problem of weakening the concept of a double category so that the unit and associativity properties hold up to higher-dimensional morphisms can be contrasted with a different approach. One might instead try to combine the notions of *bicategory* and *double category* in a different way. This is by "doubling" the notion of bicategory, in the same way that double categories did with the notion of category. Just as a double category is an internal category in **Cat**, the result would be an internal bicategory in **Bicat**.

We would like to call this a *double bicategory*: however, this term has already been used by Dominic Verity to describe the structure we will mainly be interested in. Since the former concept is also important for us in certain lemmas, and is most naturally called a double bicategory, we will refer to the latter as a *Verity double bicategory*. For more discussion of the relation between these, see section 6.

4.2. **Definition of a Double Bicategory.** The following definition of a Verity double bicategory is due to Dominic Verity ([Ver]), and is readily seen as a natural weakening of the definition of a double category. Just as the concept of *bicategory* weakens that of *2-category* by weakening the associative and unit laws, Verity double bicategories will do the same for double categories.

Definition 4. *A **Verity double bicategory*** $\mathbf{C}$ *is a structure consisting of the following data:*

- *a class of **objects*** Obj,
- ***horizontal*** *and* ***vertical bicategories*** **Hor** *and* **Ver** *having* Obj *as their objects*
- *for every square of horizontal and vertical morphisms of the form*

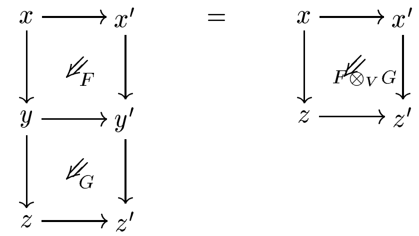
(20)
$$\begin{array}{ccc} a & \xrightarrow{h} & b \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle v'} \\ c & \xrightarrow{h'} & d \end{array}$$

a class of ***squares*** **Squ**, *with maps* $s_h, t_h : \mathbf{Squ} \to \mathrm{Mor}(\mathbf{Hor})$ *and* $s_v, t_v : \mathbf{Squ} \to \mathrm{Mor}(\mathbf{Ver})$, *satisfying an equation for each corner, namely:*

(21)
$$\begin{aligned} s(s_h) &= s(s_v) \\ t(s_h) &= s(t_v) \\ s(t_h) &= t(s_v) \\ t(t_h) &= t(t_v) \end{aligned}$$

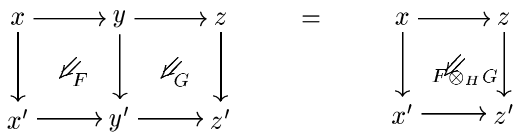
The squares should have horizontal and vertical composition operations, defining the vertical composite $F \otimes_V G$

(22)



and horizontal composite $F \otimes_H G$*:*

(23)

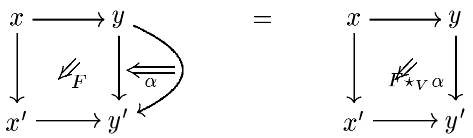


These have the usual relation to source and target maps, satisfy the interchange law

$$(F \otimes_V F') \otimes_H (G \otimes_V G') = (F \otimes_H G) \otimes_V (F' \otimes_H G') \tag{24}$$

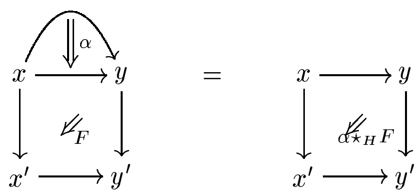
and have a left and right action by the horizontal and vertical 2-morphisms on **Squ**, *giving* $F \star_H \alpha$,

(25)



(and similarly on the left) and $F \star_V \alpha$,

(26)



The actions are compatible with composition:

$$(F \otimes_H G) \star_V \alpha = F \otimes_H (G \star_V \alpha) \tag{27}$$

(and analogously for vertical composition). They also satisfy additional compatibility conditions: the left and right actions of both vertical and horizontal 2-morphisms satisfy the "associativity" property

$$\alpha \star (S \star \beta) = (\alpha \star S) \star \beta \tag{28}$$

for both $\star_H$ or $\star_V$ in either position. Moreover, horizontal and vertical actions are independent:

$$\alpha \star_H (\beta \star_V S) = \beta \star_V (\alpha \star_H S) \tag{29}$$

and similarly for the right action.

We note that, although this definition is fairly elaborate, it is simpler than would be a similarly elementary description of a double bicategory. Indeed, in section 5.3 we that a Verity double bicategory is a special case of a double bicategory, satisfying some extra properties.

In particular, where there are combatibility conditions involving equations in this definition, such a structure would have only isomorphisms, themselves satisfying additional coherence laws. In particular, in double bicategories, the action of 2-morphisms on squares is described by strict equations, rather than being given by a definite isomorphism.

Similarly, it is possible (see [Ver] sec. 1.4) to define categories $\mathbf{Cyl_H}$ and $\mathbf{Cyl_V}$ of *cylinders* whose objects are squares, and maps are pairs of vertical (respectively, horizontal) 2-morphisms joining the vertical (resp. horizontal) source and targets of pairs of squares which share the other two sides. These are plain categories, with strict associativity and unit laws. These conditions would be weakened in a double bicategory (in which maps would include not just pairs of 2-morphisms, but also a 3-dimensional interior of the cylinder - a morphism in $2\,\mathrm{Mor}$, or 2-morphism in Mor, satisfying properties only up to a 4-dimensional 2-morphism in $2\,\mathrm{Mor}$).

However, the definition given above, despite being a more special case, having no (nontrivial) morphisms of more than 2 dimensions, contains as much structure as we need to describe our intended examples.

4.3. **An Equivalence Theorem.** There are numerous connections between double categories and bicategories (or their strict form, 2-categories). One is Ehresmann's double category of quintets, relating double categories to 2-categories: a double category by taking the squares to be 2-morphisms between composite pairs of morphisms, such as $\alpha : g' \circ f \to f' \circ g$.

Furthermore, it is well known that double categories can be made equivalent to 2-categories in three different ways. Two obvious cases are when there are only identity horizontal and vertical morphisms, respectively, so that squares simply collapse into bigons.Notice that it is also true that a double bicategory in which **Hor** is trivial (equivalently, if **Ver** is trivial) is again a bicategory. The squares become 2-morphisms in the obvious way, the action of 2-morphisms on squares then is just composition, and the composition rules for squares and bigons are the same. The result is clearly a bicategory.

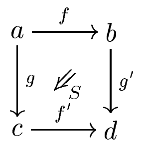
The other, less obvious, case, is when the horizontal and vertical morphisms are identified - that is, when the horizontal and vertical categories on the objects are the same. Then we again can interpret squares as bigons by composing the top and right edges, and the left and bottom edges. Introducing identity bigons completes the structure. These new bigons have a natural composition inherited from that for squares. It turns out that this yields a structure satisfying the definition of a 2-category. Here, our goal will be to show an analogous result, that a Verity double bicategory similarly gives rise to a bicategory when the horizontal and vertical bicategories are equal. We expect that the converse holds as well - but we show one direction only.

Our condition that **Hor** = **Ver** holds in our general example of double spans: both horizontal and vertical bicategories in any $2\mathrm{Span}(\mathbf{C})_0$ are just $\mathrm{Span}(\mathbf{C})$. In particular, this result will also apply to our example of cobordisms.

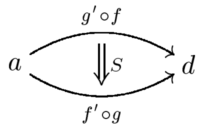
Theorem 1. *Any Verity double bicategory* $(\mathrm{Obj}, \mathbf{Hor}, \mathbf{Ver}, \mathbf{Squ}, \otimes_H, \otimes_V, \star_H, \star_V)$ *for which* **Hor** = **Ver** *produces a bicategory by taking squares to be 2-cells.*

Proof. We begin by defining the data of this bicategory, which we call **B**. Its objects and morphisms are the same as those of **Ver** (equivalently, **Hor**). We describe the 2-morphisms by observing that **B** must contain all those in **Ver** (equivalently, **Hor**), but also some others, which correspond to the squares in **Squ**.

In particular, given a square

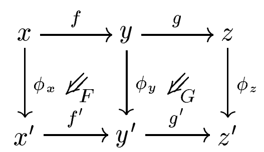
(30)


there should be a 2-morphism

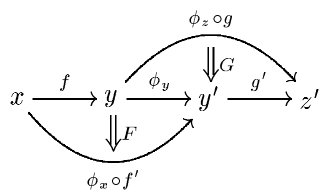
(31)


The composition of squares corresponds to either horizontal or vertical composition of 2-morphisms in **B**, and the equivalence of these is given in terms of the interchange law in a bicategory:

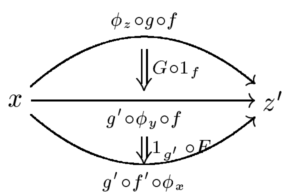
Given a composite of squares,

(32)


there will be a corresponding diagram in **B**:

(33)


Using horizontal composition with identity 2-morphisms, we can write this as a vertical composition:

(34)


So the square $F \otimes_H G$ corresponds to $(1 \circ G) \cdot (F \circ 1)$ for appropriate identities 1. Similarly, the vertical composite of $F' \otimes_V G'$ must be the same as $(1 \circ F) \cdot (G \circ 1)$. Thus, every composite of squares, which can all be built from horizontal and vertical composition, gives a corresponding composite of 2-morphisms in **B**, which are generated by those correpsonding to squares in **Squ**, subject to the relations imposed by the composition rules in a bicategory.

To show the Verity double bicategory gives a bicategory, it now suffices to show that all such 2-morphisms not already in **Ver** arise as squares (that is, the structure is closed under composition). So suppose we have any composable pair of 2-morphisms which arise from squares. If the squares have an edge in common, then we have the situation depicted above (or possibly the equivalent in the vertical direction). In this case, the composite 2-morphism corresponds exactly to the composite of squares, and the axioms for composition of squares ensure that all 2-morphisms generated this way are already in our bicategory.

If there is no edge in common, the 2-morphisms in **B** must be made composable by composition with identities. In this case, all the identities can be derived from 2-morphisms in **Ver**, or from identity squares in **Squ** (inside commuting diagrams). Clearly, any identity 2-morphism can be factored this way. Then, again, the composite 2-morphisms in **B** will correspond to the composite of all such squares and 2-morphisms in **Squ** and **Ver**. □

4.4. **Double Spans.** Now we construct a class of examples. These examples are analogous to the example of bicategories of spans, discussed in section 3.3. These span-ish examples of Verity double bicategories are closely related to a topological example similar in flavour to the topological examples of bicategories and double categories in section 3.5.

We remarked in section 3.4 that a double category is a category internal to **Cat**. In 4 we observed that Verity double bicategories can similarly be understood in terms of double bicategories (with suitable restriction to isomorphism classes at the top-dimensional level). The construction we will make here uses this idea in the particular case where all the bicategories involved are realized as Span(**C**) for some **C**. These examples are also analogous to the "profunctor-based examples" of pseudo-double categories described by Grandis and Paré [GP2]. The important example for us here is $2\mathrm{Span}(\mathbf{C})_0$ (we will see the reason for this notation shortly).

In Remark 1 we described Bénabou's demonstration that **Obj** = Span(**C**) is a bicategory. There is an analogous fact about double spans, which can be described in terms of double bicategories. These are described explicitly in section 6. We begin by describing 2Span(**C**). The Verity double bicategory described above is derived from this, as we shall show shortly.

Definition 5. 2Span(**C**) *is a double bicategoryof* ***double spans*** *in* **C***, consisting of the following:*

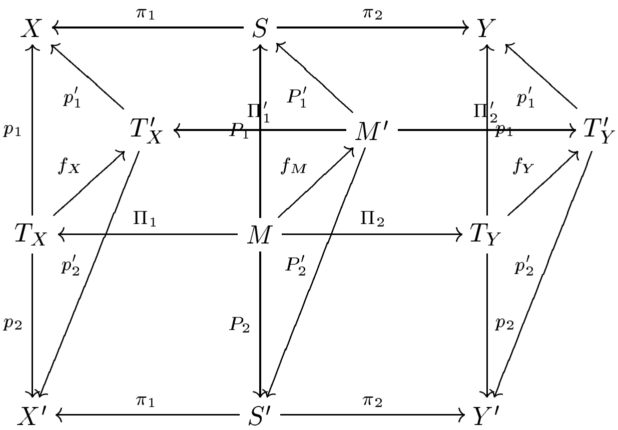
- *the bicategory of objects is* **Obj** = Span(**C**)

- *the bicategory of morphisms* **Mor** *has spans in* **C** *as objects, as morphisms commuting diagrams of the form:*

$$\begin{array}{ccccc} X & \xleftarrow{\pi_1} & S & \xrightarrow{\pi_2} & Y \\ {\scriptstyle p_1}\uparrow & & {\scriptstyle P_1}\uparrow & & \uparrow{\scriptstyle p_1} \\ T_X & \xleftarrow{\Pi_1} & M & \xrightarrow{\Pi_2} & T_Y \\ {\scriptstyle p_2}\downarrow & & {\scriptstyle P_2}\downarrow & & \downarrow{\scriptstyle p_2} \\ X' & \xleftarrow{\pi_1} & S' & \xrightarrow{\pi_2} & Y' \end{array} \tag{35}$$

- *as 2-morphisms commuting diagrams of the form:*

(36)



- *the bicategory of 2-morphisms has as objects span maps in* **C** *as in (11), as morphisms spans of span maps (as in (36), but with span maps horizontal), and as 2-morphisms span maps of span maps*

All composition operations are by pullback; source and target operations follow those for spans.

Define 2Cosp(**C**) *as* 2Span($\mathbf{C}^{\mathrm{op}}$).

In section 6 we show (lemma 3) that for any category **C** with pullbacks, 2Span(**C**) forms a double bicategory.

Remark 2. Just as 2-morphisms in **Mor** and morphisms in **2Mor** can be seen as diagrams which are "products" of a span with a map of spans, 2-morphisms in **2Mor** are given by diagrams which are "products" of horizontal and vertical span maps. These have, in either direction, four maps of spans, with objects joined by maps of spans. Composition again is by pullback in composable pairs of diagrams.

In fact, there is more structure here than we really need to describe our example of cobordisms with corners. There is another Verity double bicategory which we can derive from 2Span(**C**) by considering it only up to a certain kind of equivalence:

Definition 6. *For a category* **C** *with finite limits, the Verity double bicategory* 2Span$(\mathbf{C})_0$*, has:*

- *the objects are objects of* **C**
- *the horizontal and vertical bicategories* **Hor** = **Ver** *are equal to a sub-bicategory of* Span(**C**)*, which includes only invertible span maps*

- *the squares are isomorphism classes of commuting diagrams of the form (35)*

where two diagrams of the form (35) are isomorphic if they differ only in the middle objects, say M and M', and their maps to the edges, and if there is an isomorphism $f : M \to M'$ making the combined diagram commute.

The action of 2-morphisms α in **Hor** *and* **Ver** *on squares is by composition in diagrams of the form:*

(37)
$$\begin{array}{ccccc}
 & \swarrow^{\pi_1} & S_2 & {}^{\pi_2}\searrow & \\
 & & \uparrow{\scriptstyle \alpha} & & \\
X & \xleftarrow{\pi_1} & S_1 & \xrightarrow{\pi_2} & Y \\
{\scriptstyle p_1}\uparrow & & {\scriptstyle P_1}\uparrow & & \uparrow{\scriptstyle p_1} \\
T_X & \xleftarrow{\Pi_1} & M & \xrightarrow{\Pi_2} & T_Y \\
{\scriptstyle p_2}\downarrow & & {\scriptstyle P_2}\downarrow & & \downarrow{\scriptstyle p_2} \\
X' & \xleftarrow{\pi_1} & S' & \xrightarrow{\pi_2} & Y'
\end{array}$$

(where the resulting square is as in 35, with S_2 in place of S and $\alpha \circ P_1$ in place of P_1).

Composition (horizontal or vertical) of squares of spans is, as in $2\mathrm{Span}(\mathbf{C})$*, given by composition (by pullback) of the three spans of which the square is composed. The composition operators for diagrams of span maps are by the usual ones in* $\mathrm{Span}(\mathbf{C})$.

Define $2\mathrm{Cosp}_0(\mathbf{C})$ *as* $2\mathrm{Span}(\mathbf{C})_0(\mathbf{C}^{\mathrm{op}})$.

Remark 3. Notice that **Hor** and **Ver** as defined are indeed bicategories: eliminating all but the invertible 2-morphisms leaves a collection which is closed under composition by pullbacks.

We show more fully that this is a Verity double bicategory in theorem 2, but for now we note that the definition of horizontal and vertical composition of squares is defined on equivalence classes. One must show that this is well defined. We will get this result indirectly as a result of lemmas 3 and 4, but it is instructive to see directly how this works in $\mathrm{Span}(\mathbf{C})$.

Lemma 1. *The composition of squares in Definition 6 is well-defined.*

Proof. Suppose we have two representatives of a square, bounded by horizontal spans (S, π_1, π_2) from X to Y and (S', π_1, π_2) from X' to Y', and vertical spans (T_X, p_1, p_2) from X to X' and (T_Y, p_1, p_2) from Y to Y'. The middle objects M_1 and M_2 as in the diagram (35). If we also have a composable diagram - one which coincides along an edge (morphism in **Hor** or **Ver**) with the first, then we need to know that the pullbacks are also isomorphic (that is, represent the same composite square).

In the horizontal and vertical composition of these squares, the maps from the middle object M of the new square to the middle objects of the new sides (given by composition of spans) arise from the universal property of the pullbacks on the sides being composed (and the induced maps from M to the corners, via the maps in the spans on the other sides). Since the middle objects are defined only up to isomorphism class, so is the pullback: so the composition is well defined, since the result is again a square of the form (35). □

In section 5.3 we show that $2\mathrm{Span}(\mathbf{C})_0$ is a Verity double bicategory. For now, we will examine how cobordisms form a special topological example of this sort of Verity double bicategory.

5. Cobordisms

One of our motivations for studying Verity double bicategories is to provide the right formal structure for some special examples. The examples we have in mind are higher categories of cobordisms. The objects in these categories are manifolds of some dimension, say k. In this case, the morphisms are $(k+1)$-dimensional cobordisms between these manifolds: that is, manifolds with boundary, such that the boundary decomposes into two components, with one component as the source, and one as the target. The 2-cells are $(k+2)$-dimensional cobordisms between $(k+1)$-dimensional cobordisms: these can be seen as manifolds with corners, where the corners are the k-dimensional objects.

We could continue building a ladder in which the $j+1$-cells are cobordisms between the j-cells, which are cobordisms between the $(j-1)$-cells, but two levels is enough to give a Verity double bicategory. We will see that these can be construed using the double span construction of section 4.4.

5.1. Collars on Manifolds with Corners.

Here we will use our construction of a Verity double bicategory $2\mathrm{Span}(\mathbf{C})$ from section 4.4 in order to show an example of a double bicategory of cobordisms with corners, starting with $\mathbf{C}$, a certain category of smooth spaces. To begin with, we recall that a smooth manifold with corners is a topological manifold with boundary, together with a certain kind of C^∞ structure. In particular, we need a maximal compatible set of coordinate charts $\phi : \Omega \to [0,\infty)^n$ (where ϕ_1, ϕ_2 are compatible if $\phi_2 \circ \phi_1^{-1}$ is a diffeomorphism). The fact that the maps are into the positive sector of $\mathbb{R}^n$ distinguishes a manifold with corners from a manifold.

Jänich [Jan] introduces the notion of $\langle n \rangle$-manifold, reviewed by Laures [Laur]. This is build on a manifold with faces:

Definition 7. *A face of a manifold with corners is the closure of some connected component of the set of points with just one zero component in any coordinate chart). An $\langle n \rangle$-manifold is a manifold with faces together with an n-tuple $(\partial_0 M, \dots, \partial_{n-1} M)$ of faces of M, such that*

- $\partial_0 M \cup \dots \partial_{n-1} M = \partial M$
- $\partial_i M \cap \partial_j M$ *is a face of* $\partial_i M$ *and* $\partial_j M$

The case we will be interested in here is the case of $\langle 2 \rangle$-manifolds. In this notation, a $\langle 0 \rangle$-manifold is just a manifold without boundary, a $\langle 1 \rangle$-manifold is a manifold with boundary, and a $\langle 2 \rangle$-manifold is a manifold with corners whose boundary decomposes into two components (of codimension 1), whose intersections form the corners (of codimension 2). We can think of $\partial_0 M$ and $\partial_1 M$ as the "horizontal" and "vertical" part of the boundary of M.

Example 3. Let M be the solid 3-dimensional illustrated in figure 1. The boundary decomposes into 2-dimensional manifolds with boundary. Denote by $\partial_0 M$ the boundary component consisting of the top and bottom surfaces, and $\partial_1 M$ be the remaining boundary component (a topological annulus).

In this case, $\partial_0 M$ is the disjoint union of the manifolds with corners S (two annuli) and S' (topologically a three punctured sphere); $\partial_1 M$ is the disjoint union of two components, T_X (which is topologically a three-punctured sphere) and T_Y (topologically a four-punctured torus).

Then we have $\partial_0 M \cup \partial_1 M = \partial M$. Also, $\partial_0 M \cap \partial_1 M$ is a 1-dimensional manifold without boundary, which is a face of both $\partial_0 M$ and $\partial_1 M$ (in fact, the shared boundary). In particular, it is the disjoint union $X \cup Y \cup X' \cup Y'$.

We have described a Verity double bicategory formed from all cobordisms with corners in a category obtained by co-completing that of all such cobordisms, so that all pushouts exist. The problem with this is that the pushout of two cobordisms M_1 and M_2 over a submanifold S included in both by maps $S \stackrel{i_1}{\to} M_1$ and $S \stackrel{i_2}{\to} M_2$ may not be a cobordism. If the submanifolds are not on the boundaries, certainly the result may not even be a manifold: for instance, two line segments with a common point in the interior. So to get a Verity double bicategory in which the morphisms are smooth manifolds with boundary, certainly we can only consider the case where we compose two cobordisms by a pushout along shared submanifolds S which are components of the boundary of both M_1 and M_2.

However, even if the common submanifold is at the boundary, there is no guarantee that the result of the pushout will be a smooth manifold. In particular, for a point $x \in S$, there will be a neighborhood U of x which restricts to $U_1 \subset M_1$ and $U_2 \subset M_2$ with smooth maps $\phi_i : U_i \to [0,\infty)^n$ with $\phi_i(x)$ on the boundary of $[0,\infty)^n$ with exactly one coordinate equal to 0. One can easily combine these to give a homeomorphism $\phi : U \to \mathbb{R}^n$, but this will not necessarily be a diffeomorphism along the boundary S.

To solve this problem, we use the *collaring theorem*: For any smooth manifold with boundary M, ∂M has a *collar*: an embedding $f : \partial M \times [0,\infty) \to M$, with $(x,0) \mapsto x$ for $x \in \partial M$. This is a well-known result (for a proof, see e.g. [Hir], sec. 4.6). It is an easy corrolary of this usual form that we can choose to use the interval $[0,1]$ in place of $[0,\infty)$ here.

In ([Laur], Lemma 2.1.6), Gerd Laures describes a generalization of this theorem to $\langle n \rangle$-manifolds, so that for any $\langle n \rangle$-manifold M, there is an n-dimensional cubical diagram (*$\langle n \rangle$-diagram*) of embeddings of cornered neighborhoods of the faces. It is then standard that one can compose two smooth cobordisms with corners, equipped with such smooth collars, by gluing along S. The composite is then the topological pushout of the two inclusions. Along the collars of S in M_1 and M_2, charts $\phi_i : U_i \to [0,\infty)^n$ are equivalent to charts into $\mathbb{R}^{n-1} \times [0,\infty)$, and since the the composite has a smooth structure defined up to a diffeomorphism[1] which is the identity along S.

[1]Note that the precise smooth structure on this cobordism depends on the collar which is chosen - but that there is always such a choice, and the resulting composites are all equivalent up to diffeomorphism. That is, they are equivalent up to a 2-morphism in the bicategory. So strictly speaking, the composition map is not a functor but an anafunctor. It is common to disregard this issue, since one can always define a functor from an anafunctor by using the axiom of choice. This is somewhat unsatisfactory, since it does not generalize to the case where our categories are over a base in which the axiom of choice does not hold, but this is not a problem in our example.

5.2. **Cobordisms with Corners.** Suppose we take the category **Man** whose objects are smooth manifolds with corners and whose morphisms are smooth maps. Naively, would would like to use the cospan construction from section 4.4, we obtain a Verity double bicategory 2Cosp(**Man**). While this approach will work with the category **Top**, however, it will not work with **Man** since this does not have all colimits. In particular, given two smooth manifolds with boundary, we can glue them along their boundaries in non-smooth ways, so to ensure that the pushout exists in **Man** we need to specify a smoothness condition. We describe in section 5.1 how to find a subcategory with finite colimits in which all objects, morphisms, and squares are indeed manifolds with (possibly empty) corners. This requires using collars on the boundaries and corners.

For each n, we define a Verity double bicategory within **Man**, which we will call $\mathbf{nCob_2}$:

Definition 8. *The Verity double bicategory* $\mathbf{nCob_2}$ *is given by the following data:*

- *The objects of* $\mathbf{nCob_2}$ *are of the form* $P = \hat{P} \times I^2$ *where* $\hat{P}$ *may be any* $(n-2)$ *manifolds without boundary and* $I = [0,1]$*.*
- *The horizontal and vertical bicategories of* $\mathbf{nCob_2}$ *have*
  - *objects: as above*
  - *morphisms: cospans* $P_1 \xrightarrow{i_1} S \xleftarrow{i_2} P_2$ *where* $S = \hat{S} \times I$ *and* $\hat{S}$ *may be any of those cospans of* $(n-1)$*-dimensional manifolds-with-boundary which are cobordisms with collars such that the* $\hat{P_i} \times I$ *are objects, the maps are injections into* S*, a manifold with boundary, such that* $i_1(P_1) \cup i_2(P_2) = \partial S \times I$*,* $i_1(P_1) \cap i_2(P_2) = \emptyset$*,*
  - *2-morphisms: cospan maps which are diffeomorphisms of the form* $f \times \mathrm{id} : T \times [0,1] \to T' \times [0,1]$ *where* T *and* T' *have a common boundary, and* f *is a diffeomorphism*$T \to T'$ *compatible with the source and target maps - i.e. fixing the collar.*

  where the source of a cobordism S *consists of the collection of components of* $\partial S \times I$ *for which the image of* $(x,0)$ *lies on the boundary for* $x \in \partial S$*, and the target has the image of* $(x,1)$ *on the boundary*
- *squares: diffeomorphism classes of* n*-dimensional manifolds* M *with corners satisfying the properties of* M *in the diagram of equation (35), where isomorphisms are diffeomorphisms preserving the boundary*
- *the action of the diffeomorphisms on the "squares" (classes of manifolds* M*) is given by composition of diffeomorphisms of the boundary cobordisms with the injection maps of the boundary* M

The source and target objects of any cobordism are the collars, embedded in the cobordism in such a way that the source object $P = \hat{P} \times I^2$ *is embedded in the cobordism* $S = \hat{S} \times I$ *by a map which is the identity on* I *taking the first interval in the object to the interval for a horizontal morphism, and the second to the interval for a vertical morphism. The same condition distinguishing source and target applies as above.*

Composition of squares works as in $2\mathrm{Span}(\mathbf{C})_0$*.*

We will see that $\mathbf{nCob_2}$ is a Verity double bicategory in section 5.3, but for now it suffices to note that since it is composed of double cospans, we can hope to define composition to be just that in the Verity double bicategory$2\mathrm{Span}(\mathbf{C})_0$ where

C is the category of manifolds with corners. The proof that this is a Verity double bicategory will entail showing that $\mathbf{nCob}_2$ is closed under this composition.

Lemma 2. *Composing horizontal morphisms in* $\mathbf{nCob}_2$ *this way produces another horizontal morphism in* $\mathbf{nCob}_2$*. Similarly, composition of vertical morphisms produces a vertical morphism, and composition of squares produces another square.*

Proof. The horizontal and vertical morphisms are products of the interval I with $\langle 1 \rangle$-manifolds, whose boundary is $\partial_0 S$), equipped with collars. Suppose we are given two such cobordisms S_1 and S_2, and an identification of the source of S_2 with the target of S_1 (say this is $P = \hat{P} \times I$). Then the composite $S_2 \circ S_1$ is topologically the pushout of S_1 and S_2 over P. Now, P is smoothly embedded in S_1 and S_2, and any point in the pushout will be in the interior of either S_1 or S_2 since for any point on $\hat{P}$ each end of the interval I occurs as the boundary of only one of the two cobordisms. So the result is smooth. Thus, **2Cob** is closed under such composition of morphisms.

The same argument holds for squares, since it holds for any representative of the equivalence class of some manifold with corners, M, and the differentiable structure will be the same, since we consider equivalence up to diffeomorphisms which preserve the collar exactly. □

This establishes that composition in $\mathbf{nCob}_2$ is well defined, and composites are again cobordisms in $\mathbf{nCob}_2$. We show that it is a Verity double bicategoryin section 5.3.

Example 4. We can represent a typical manifestation of the diagram (35) as in figure 1.

Consider how this picture is related to (35). In the figure, we have $n = 3$, so the objects are (compact, oriented) 1-dimensional manifolds, thickened by taking a product with I^2. X (top, solid lines) and Y (top, dotted lines) are both isomorphic to $(S^1 \cup S^1) \times I^2$, while X' and Y' (bottom, solid and dotted respectively) are both isomorphic to $S^1 \times I^2$.

The horizontal morphisms are (thickened) cobordisms S, and S', which are a pair of thickened annuli and a two-holed disk, respectively, with the evident injection maps from the objects X, Y, X', Y'. The vertical morphisms are the thickened cobordisms T_X and T_Y. In this example, T_X happens to be of the same form as S' (a two-holed disk), and has inclusion maps from X and X', the two components of its boundary, as the "source" and "target" maps. T_Y is homotopy equivalent to a four-punctured torus, where the four punctures are the components of its boundary - two circles in Y and two in Y', which again have the obvious inclusion maps. Reading from top to bottom, we can describe T_Y as the story of two (thick) circles which join into one circle, then split apart, then rejoin, and finally split apart again.

Finally, the "square" in this picture is the manifold with corners, M, whose boundary has four components, $S, S', T_X, \text{and} T_Y$, and which has corners precisely along the boundaries of these manifolds - whose components are divided between the objects X, Y, X', Y'. The embeddings of these thickened manifolds and cobordisms gives a specific way to equip M with collars.

Given any of the horizontal or vertical morphisms (thickened cobordisms S, S', T_X and T_Y), a 2-morphism would be a diffeomorphism to some other cobordism

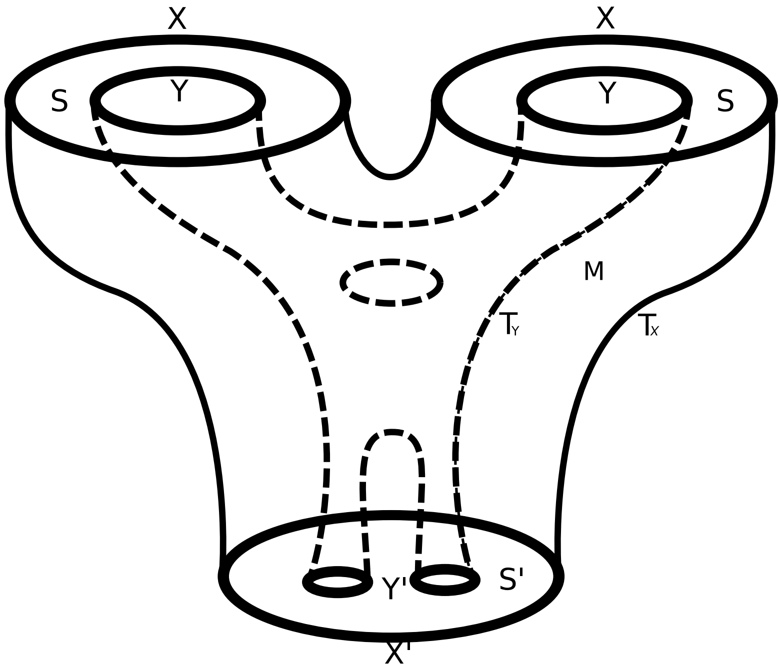


FIGURE 4. A Square in $\mathbf{nCob_2}$ (Thickened Lines Denote Collars)

equipped with maps from the same objects (boundary components), which fixes the collar on that cobordism - that is, fixes the embedded object. Such a diffeomorphism is necessarily a homeomorphism, so topologically the picture will be similar after the action of such a 2-morphism - but we would consider two such cobordisms as separate morphisms in **Hor** or **Ver**.

Remark 4. We note the resemblance between this example and $\mathrm{Path}(S)_2$ and $\mathbf{H}(S)$ defined previously. In those cases, we are considering manifolds embedded in a topological space S, and only a low-dimensional special case (the square $[0,1]\times[0,1]$ is a manifold with corners). Instead of homotopies, which make sense only for embedded spaces, $\mathbf{nCob_2}$ has diffeomorphisms. However, in both cases, we consider the squares to be *isomorphism classes* of a certain kind of top-dimensional object (homotopies or cobordisms). This eliminates the need to define morphisms or cells in our category of dimension higher than 2. We may omit this restriction if we move to a more general definition of double bicategory, as described in section 6.

5.3. **Main Theorem.** Now we want to show that cobordisms of cobordisms form a Verity double bicategory under the composition operations we have described. We will do this by first showing a more general result, including $2\mathrm{Span}(\mathbf{C})_0$ for any category $\mathbf{C}$ with finite limits, and showing how a Verity double bicategory is an internal bicategory in **Bicat**, and of a special kind which can be obtained by precisely the reduction to isomorphism classes and restriction to particular spans which we perform in defining $\mathbf{nCob_2}$.

We use lemmas 3 and 4, proved in section 6 to show the following:

Theorem 2. *If* $\mathbf{C}$ *is a category with finite limits, then* $2\mathrm{Span}(\mathbf{C})_0$ *is a Verity double bicategory. If* $\mathbf{C}$ *has finite colimits, then* $2\mathrm{Cosp}_0(\mathbf{C})$ *is a Verity double bicategory.*

Proof. For any cocomplete category $\mathbf{C}$, $2\mathrm{Span}(\mathbf{C})$ as defined above forms a double bicategory (Lemma 3). Then in the construction of $2\mathrm{Span}(\mathbf{C})_0$, we take isomorphism classes of double spans as the squares - that is, 2-isomorphism classes of morphisms in **Mor** in the double bicategory, where the 2-isomorphisms are invertible span maps, in both horizontal and vertical directions. We also restrict to invertible span maps in the horizontal and vertical bicategories.

We are then effectively discarding all morphisms and 2-morphisms in **2Mor**, and the 2-morphisms in **Mor** except for the invertible ones. In particular, there may be "squares" of the form (35) in $2\mathrm{Span}(\mathbf{C})$ with non-invertible maps joining their middle objects M - but we have ignored these, and also ignore non-invertible span maps in the bicategories on the edges. Thus, we consider no diagrams of the form (36) except for invertible ones - in which case, the middle objects M and M' are representatives of the same isomorphism class. Similar reasoning applies to the 2-morphisms in **2Mor**.

The resulting structure we get from discarding these will again be a double bicategory. In particular, the new **Mor** and **2Mor** will be bicategories, since they are, respectively, just a category and a set made into a discrete bicategory by adding identities. On the other hand, for the composition, source and target maps to be bifunctors amounts to saying that the structures built from the objects, morphisms, and 2-cells respectively are again bicategories, since the composition, source, and target maps satisfy the usual axioms. But the same argument applies to those built from the morphisms and 2-cells as within **Mor** and **2Mor**. So we have a double bicategory.

Next we show that the horizontal and vertical action conditions (definition 11) hold in $2\mathrm{Span}(\mathbf{C})$. A square in $2\mathrm{Span}(\mathbf{C})$ is a diagram of the form (35), and a 2-cell is a map of spans. Given a square M_1 and 2-cell α with compatible source and targets as in the action conditions, we have a diagram of the form shown in (37). Here, M_1 is the square diagram at the bottom, whose top row is the span containing S_1. The 2-cell α is the span map including the arrow $\alpha : S_1 \to S_2$. There is a unique square built using the same objects as M_1 except using the span containing S_2 as the top row. The map to S_2 from M is then $\alpha \circ P_1$.

To satisfy the action condition, we want this square M_2, which is the candidate for $M_1 \star_V \alpha$, to be unique. But suppose there were another M_2' with a map to S_2. Since we are in $2\mathrm{Span}(\mathbf{C})_0$, α must be invertible, which would give a map from M_2' to S_1. We then find that M_2' and M_2 are representatives of the same isomorphism class - so in fact this is the same square. That is, there is a unique morphism in **2Mor** taking M_1 to M_2 (a diagram of the form 36, oriented vertically) with invertible span maps in the middle and bottom rows. This is the unique filler for the pillow diagram required by definition 11.

The argument that $2\mathrm{Span}(\mathbf{C})_0$ satisfies the action compatibility condition is similar.

So $2\mathrm{Span}(\mathbf{C})_0$ is a double bicategory in which, there there is at most one unique morphism in **Mor**, and at most unique morphisms and 2-morphisms in **2Mor**, for any specified source and target, and the horizontal and vertical action conditions hold. So $2\mathrm{Span}(\mathbf{C})_0$ can be interpreted as a Verity double bicategory (Lemma 4).

The case where we begin with a category **C** with finite colimits and use cospans can be reduced to this case, by taking $\mathbf{C}^{\mathrm{op}}$. □

The argument that double spans or double cospans form a Verity double bicategory can be slightly modified to show the same about cobordisms with corners. We note that there are two differences. First, the category of manifolds with corners does not have all finite colimits. Second, we are not dealing with all double cospans of manifolds with corners, so $\mathbf{nCob}_2$ is not $2\mathrm{Span}(\mathbf{C})_0$ for any **C**. In fact, the second difference is what allows us to deal with the first.

Theorem 3. $\mathbf{nCob}_2$ *is a Verity double bicategory.*

Proof. First, recall that objects in $\mathbf{nCob}_2$ are manifolds with corners of the form $P = \hat{P} \times I^2$ for some manifold $\hat{P}$, and notice that both horizontal and vertical morphisms are cospans. In general, if we have two cospans in the category of manifolds with corners sharing a common object, we cannot take a pullback and get a manifold with corners. However, we are only considering a subset of all possible spans of smooth manifolds with corners, all all those we consider have pullbacks which are again smooth manifolds with corners (lemma 2).

In particular, since composition of squares is as in $2\mathrm{Span}(\mathbf{C})_0$, before taking diffeomorphism classes of manifolds M in $\mathbf{nCob}_2$, we would again get a double bicategory made from cobordisms with corners, together with the embeddings used in its cospans, and collar-fixing diffeomorphisms. This is shown by arguments identical to those of lemma 3.

When we reduce to diffeomorphism classes of these manifolds, then just as in the proof of 2, we can cut down this double bicategory to a structure, and the result will satisfy the horizontal and vertical action conditions, giving a Verity double bicategory, since it satisfies the conditions of lemma 4.

So in fact, by the same arguments as in these other cases, $\mathbf{nCob}_2$ is a Verity double bicategory. □

6. Internal Bicategories in **Bicat**

6.1. **Introduction.** We rely on the notion of a bicategory *internal* to **Bicat** at several points in this paper. Here we present a more precise definition of this concept, and in lemmas 3 and 4 we use it to show that examples having properties like those of $2\mathrm{Span}(\mathbf{C})_0$ (definition 6) give double bicategories in the sense of Verity. These lemmas were used in the proofs Theorems 2 and 3.

To begin with, we remark that the theory of bicategories, **Th(Bicat)** is more complicated than that for categories. However as with **Th(Cat)**, it will be a category with objects Obj, **Mor** and **2Mor**, and having all equalizers, pullbacks. To our knowledge, a model of **Th(Bicat)** in **Bicat** has not been explicitly described before. We could treat Obj as a horizontal bicategory, and the objects of Obj, **Mor** and **2Mor** as forming a vertical bicategory, but we note that diagrammatic representation of, for instance, 2-morphisms in **2Mor** would require a 4-dimensional diagram element. The comparison can be seen by contrasting tables 1 and 2. The axioms satisfied by such a structure are rather more unwieldy than either a bicategory or a double category, but they provide some coherence to the axioms for a Verity double bicategory, as shown in definition 4, as we shall see in section 6.4.

We start by describing how to obtain a double bicategory.

6.2. **The Theory of Bicategories.** We described in section 3.4 how a double category may be seen as a category internal to **Cat**. To put it another way, it a model of **Th(Cat)**, the theory of categories, in **Cat**, which is a limit-preserving functor from **Th(Cat)** into **Cat**. We did not make a special point of the fact, but this is a *strict* model. A weak model would satisfy the category axioms such as composition only up to a 2-morphism in **Cat** - that is, up to natural transformation. So, for instance, the pullback (15) would be a weak pullback, so that instead of satisfying $t \circ c_1 = s \circ c_2$, there would only be a natural transformation relating $t \circ c_1$ and $s \circ c_2$. Such a weak model is the most general kind of model available in **Cat**, but double categories arise as strict models.

So here we note that we are thinking of **Bicat** as a mere category, and that we are speaking of *strict* internanl bicategories. In particular, the most natural structure for **Bicat** is that of a tricategory: it has objects which are bicategories, morphisms which are bifunctors between bicategories, 2-morphisms which are natural transformations between bifunctors, and 3-morphisms which are "modifications" of such transformations. Indeed, **Bicat** is the standard example of a tricategory, just as **Cat** is the standard example of a bicategory. But we ignore the tricategorical structure for our purposes.

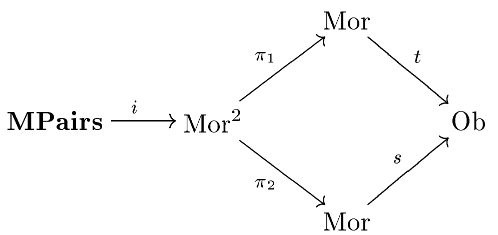
Similarly, we only consider strict models of the theory of bicategories, **Th(Bicat)** in **Bicat**. That is, a *strict* functor from the category **Th(Bicat)** into **Bicat** (a tricategory). Thus, equations in the model are mapped to equations (not isomorphisms) in **Bicat**. This is what we will call a double bicategory.

Before we can say explicitly what this means, we must explicitly describe **Th(Bicat)** as we did for **Th(Cat)** in section 3.4.

Definition 9. *The theory of bicategories is the category (with finite limits)* **Th(Bicat)** *given by the following data:*

- *Objects* Ob*,* Mor*,* 2Mor
- *Morphisms* $s, t : \mathrm{Ob} \to \mathrm{Mor}$ *and* $s, t : \mathrm{Mor} \to \mathrm{2Mor}$
- ***composition*** *maps* $\circ : \mathrm{MPairs} \to \mathbf{Mor}$ *and* $\cdot : \mathrm{BPairs} \to \mathbf{2Mor}$*, satisfying the interchange law (4), where* $\mathrm{MPairs} = \mathrm{Mor} \times_{\mathrm{Ob}} \mathrm{Mor}$ *and* $\mathbf{BPairs} = \mathrm{2Mor} \times_{\mathrm{Mor}} \mathrm{2Mor}$ *are equalizers of diagrams of the form:*

(38)



and similarly for opnameBPairs.

- *the* ***associator*** *map* $a : \mathrm{Triples} \to \mathbf{2Mor}$*, where* $\mathrm{Triples} = \times_{\mathrm{Ob}} \mathrm{Mor} \times_{\mathrm{Ob}} \mathrm{Mor}$ *is the equalizer of a similar diagram for involving* Mor^3*, such that a satisfies* $s(a(f,g,h)) = (f \circ g) \circ h$ *and* $t(a(f,g,h)) = f \circ (g \circ h)$
- ***unitors*** $l, r : \mathrm{Ob} \to \mathrm{Mor}$ *with* $s \circ l = t \circ l = \mathrm{id}_{\mathrm{Ob}}$ *and* $s \circ r = t \circ r = \mathrm{id}_{\mathrm{Ob}}$

This data is subject to the conditions that the associator is subject to the Pentagon identity, and the unitors obey certain unitor laws.

Remark 5. The Pentagon identity is shown in (8) and for a model of **Th(Bicat)** in **Sets**), where we can specify elements of Mor, but the general relations - that the composites on each side of the diagram are equal - hold in general. These are built from composable quadruples of morphisms and composition as indicated in the labels. Similar remarks apply to the unitor laws shown in (9).

So we have the following:

Definition 10. *A double bicategory consists of:*

- *bicategories* **Obj** *of* ***objects***, **Mor** *of* ***morphisms***, **2Mor** *of* ***2-morphisms***
- ***source*** *and* ***target*** *maps* $s, t : \mathbf{Mor} \to \mathbf{Obj}$ *and* $s, t : \mathbf{2Mor} \to \mathbf{Mor}$
- *partially defined* ***composition*** *functors* $\circ : \mathbf{Mor}^2 \to \mathbf{Mor}$ *and* $\cdot : \mathbf{2Mor}^2 \to \mathbf{2Mor}$*, satisfying the interchange law (4)*
- *partially defined* ***associator*** $a : \mathbf{Mor}^3 \to \mathbf{2Mor}$ *with* $s(a(f,g,h)) = (f \circ g) \circ h$ *and* $t(a(f,g,h)) = f \circ (g \circ h)$
- *partially defined* ***unitors*** $l, r : \mathbf{Obj} \to \mathbf{Mor}$ *with* $s(l(x)) = t(l(x)) = x$ *and* $s(r(x)) = t(r(x)) = x$

All the partially defined functors are defined for composable pairs or triples, for which sorce and target maps coincide in the obvious ways. The associator should satisfy the pentagon identity (8), and the unitors should satisfy the unitor laws (9).

With this definition in mind, we can remember Bénabou's classic example of a bicategory, that of spans. There is an analogous example here, namely *double spans*.

6.3. **The Double Span Example.** Section 4.4 describes a structure of double spans in a category **C** with pullbacks, which we denoted 2Span(**C**). The next lemma shows how this is the example we want:

Lemma 3. *For any category* **C** *with pullbacks,* 2Span(**C**) *forms a double bicategory.*

Proof. **Mor** and **2Mor** are bicategories since the composition functors act just like composition in Span(**C**) in each column, and therefore satisfies the same axioms.

Since the horizontal and vertical directions are symmetric, we can construct functors between **Obj**, **Mor**, and **2Mor** with the properties of a bicategory simply by using the same constructions that turn each into a bicategory. In particular, the source and target maps from **Mor** to **Obj** and from **2Mor** to **Mor** are the obvious maps giving the ranges of the projection maps in the diagrams (35). The partially defined (horizontal) composition maps $\circ : \mathbf{Mor}^2 \to \mathbf{Mor}$ and $\otimes_H : \mathbf{2Mor}^2 \to \mathbf{2Mor}$ are defined by taking pullbacks of diagrams in **C**, which exist for any composable pairs of diagrams because **C** has finite limits. They are functorial since they are independent of composition in the horizontal direction. The associator for composition of morphisms is given in the pullback construction.

To see that this construction gives a double bicategory, we note that **Obj**, **Mor**, and **2Mor** as defined above are indeed bicategories. **Obj**, because Span(**C**) is a bicategory. **Mor** and **2Mor** because the morphism and 2-morphism maps from the composition, associator, and other functors required for an double bicategory give these the structure of bicategories as well.

Moreover, the composition functors satisfy the properties of a bicategory for just the same reason that composition of spans does (since each of the three maps involved are given by this kind of construction). Thus, we have a double bicategory. □

Remark 6. Lemma 3 suggests one direction of generalization for double bicategories, to "n-tuple bicategories" for any n. We can extend the notion of double spans to arbitrarily high dimension. In section 8.2, we discuss in more detail how this possibility might work.

6.4. **Decategorification.** Our motivation for showing lemma 3 is to get a Verity double bicategory of cobordisms as a special example of a Verity double bicategory of double spans in suitable categories **C**. To get this, we need to define conditions which allow the action of 2-cells upon squares. It is helpful, in trying to understand what these are, to consider a "lower dimensional" example of a similar process.

In a double category, thought of as an internal category in **Cat**, we have data of four sorts, as shown in Table 1.

	Obj	**Mor**
Objects	$\bullet^x$	$\bullet \xrightarrow{f} \bullet$
Morphisms	$\begin{array}{c}\bullet\\ \downarrow g\\ \bullet\end{array}$	$\begin{array}{ccc}\bullet & \longrightarrow & \bullet\\ \downarrow & \swarrow_F & \downarrow\\ \bullet & \longrightarrow & \bullet\end{array}$

TABLE 1. Data of a Double Category

That is, a double category **DC** has categories **Obj** of objects and **Mor** of morphisms. The first column of the table shows the data of **Obj**: its objects are the objects of **DC**; its morphisms are the *vertical* morphisms. The second column shows the data of **Mor**: its objects are the *horizontal* morphisms of **DC**; its morphisms are the squares of **DC**.

Remark 7. The kind of "decategorification" we will want to do to obtain Verity double bicategories has an analog in the case of double categories. Namely, there is a condition we can impose which effectively turns the double category into a category, where the horizontal and vertical morphisms are composable, and the squares can be ignored. The sort of condition involved is similar to the *horn-filling conditions* introduced by Ross Street [Str] in his first introduction of the idea of weak ω-categories. In that case, all morphisms correspond to simplicial sets, and a horn filling condition is one which says that, for a given hollow simplex with just one face (morphism) missing from the boundary, there will be a morphism to fill that face, and a "filler" for the inside of the simplex, making the whole commute. A restricted horn-filling condition demands that this is possible for some class of candidate simplices.

For a double category, morphisms can be edges or squares, rather than n-simplices, but we can define the following "filler" condition: given any pair (f, g) of a horizontal and vertical morphism where the target object of f is the source object of g, there will be a unique pair $(h, \star)$ consisting of a unique horizontal morphism

h and unique invertible square $\star$ making the following diagram commute:

(39)
$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ {\scriptstyle h}\downarrow & \swarrow_{\star} & \downarrow{\scriptstyle g} \\ z & \xrightarrow[1_z]{} & z \end{array}$$

and similarly when the source of f is the target of g. Notice that taking f or g to be the identity in these cases implies F is the identity.

If, furthermore, there are no other interesting squares, then this double category can be seen as just a category. In that case, the unique h can just be interpreted as the composite of f and g and $\star$ as the process of composition. So we will use the notation $g \circ f$ instead of h in this situation.

To see that this defines a composition operation, we need to observe that composition defined using these fillers agrees with the usual composition in the horizontal or vertical categories, is associative, etc. For example, given morphisms as in the diagram:

$$\begin{array}{ccccc} w & \xrightarrow{f} & x & \xrightarrow{f'} & y \\ & & & & \downarrow{\scriptstyle g} \\ z & \xrightarrow[1_z]{} & z & \xrightarrow[1_z]{} & z \end{array}$$

there are two ways to use the unique-filler principle to fill this rectangle. One way is to first compose the pairs of horizontal morphisms on the top and bottom, then fill the resulting square. The square we get is unique, and the morphism is denoted $g \circ (f' \circ f)$. The second way is to first fill the right-hand square, and then using the unique morphism we call $g \circ f'$, we get another square on the left hand side, which our principle allows us to fill as well. The square is unique, and the resulting morphism is called $(g \circ f') \circ f$. Composing the two squares obtained this way must give the square obtained the other way, since both make the diagram commute, and both are unique. So we have:

$$\begin{array}{ccccc} w & \xrightarrow{f} & x & \xrightarrow{f'} & y \\ {\scriptstyle (g\circ f')\circ f}\downarrow & \swarrow_{\star} & \downarrow{\scriptstyle g\circ f'} \swarrow_{\star} & & \downarrow{\scriptstyle g} \\ z & \xrightarrow[1_z]{} & z & \xrightarrow[1_z]{} & z \end{array} \quad = \quad \begin{array}{ccc} w & \xrightarrow{f'\circ f} & y \\ {\scriptstyle g\circ(f'\circ f)}\downarrow & \swarrow_{\star} & \downarrow{\scriptstyle g} \\ z & \xrightarrow[1_z]{} & z \end{array}$$

So in fact we can "decategorify" a double category satisfying the unique filler condition, and treat it as if it were a mere category with horizontal and vertical morphisms equivalent. The composition between horizontal and vertical morphisms is given by the filler: given one of each, we can find a square of the required kind, by taking the third side to be an identity.

Remark 8. Note that our condition does not give a square for every possible combination of morphisms which might form its sources and targets. In particular, there must be an identity morphism - on the bottom in the example shown. If that identity could be any morphism h, then by choosing f and g to be identities, this would imply that every morphism must be invertible (at least weakly), since there must then be an h^{-1} with $h^{-1} \circ h$ isomorphic to the identity. When a filler square does exist, and we consider **DB** as a category **C**, the filler square indicates there is a commuting square in **C**: we think of it as the identity between the composites along the upper right and lower left.

The decategorification of a double bicategory to give a Verity double bicategory is similar, except that whereas with a double category we were cutting down only the squares (the lower-right quadrant of Table 1. We need to do more with a double bicategory, since there are more sorts of data, but they fall into a similar arrangement, as shown in Table 2.

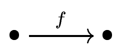
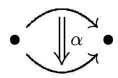
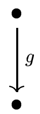
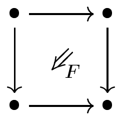
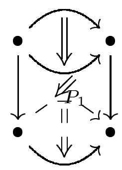
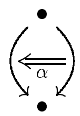
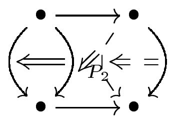
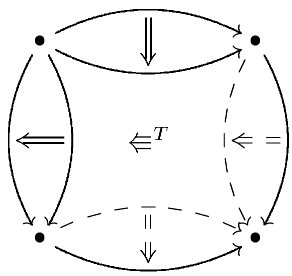
	Obj	**Mor**	**2Mor**
Objects			
Morphisms			
2-Cells			

TABLE 2. The data of a double bicategory

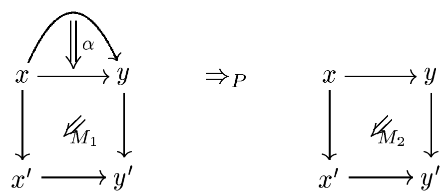
Remark 9. This shows the data of the bicategories **Obj**, **Mor**, and **2Mor**, each of which has objects, morphisms, and 2-cells. Note that the morphisms in the three entries in the lower right hand corner - 2-cells in **Mor**, and morphisms and 2-cells in **2Mor** - are not 2-dimensional. The 2-cells in **Mor** and morphisms in **2Mor** are the three-dimensional "filling" inside the illustrated cylinders, which each have two square faces and two bigonal faces.

The 2-cells in **2Mor** should be drawn 4-dimensionally. The picture illustrated can be thought of as taking both square faces of one cylinder P_1 to those of another, P_2, by means of two other cylinders (S_1 and S_2, say), in such a way that P_1 and P_2 share their bigonal faces. This description works whether we consider the P_i to be horizontal and the S_j vertical, or vice versa. These describe the "frame" of this sort of morphism: the "filling" is the 4-dimensional track taking P_1 to P_2, or equivalently, S_1 to S_2 (just as a square in a double category can be read horizontally or vertically). Not all relevant parts of the diagram have been labelled here, for clarity.

Next we want to describe a condition similar to that we gave which made it possible to think of a double category as a category. In that case, we got a condition which effectively allowed us to treat any square as an identity, so that we only had objects and morphisms. Here, we want a condition which lets us throw away the three entries of table 2 in the bottom right. This condition, when satisfied, should allow us to treat a double bicategory as a Verity double bicategory. It comes in two parts:

Definition 11. *We say that a double bicategory satisfies the* ***vertical action condition*** *if, for any morphism* $M_1 \in \mathbf{Mor}$ *and 2-morphism* $\alpha \in \mathbf{Obj}$ *such that* $s(M_1) = t(\alpha)$*, there is a morphism* $M_2 \in \mathbf{Mor}$ *and 2-morphism* $P \in \mathbf{Mor}$ *such that* P *fills the "pillow diagram":*

(40)



where M_2 *is the back face of this diagram, and the 2-morphism in* $\mathbf{Obj}$ *at the bottom is the identity.*

An double bicategory satisfies the ***horizontal action condition*** *if for any morphism* $M_1 \in \mathbf{Mor}$ *and object* α *in* $\mathbf{2Mor}$ *with* $s(M_1) = t(\alpha$ *there is a morphism* $M_2 \in \mathbf{Mor}$ *and morphism* $P \in \mathbf{2Mor}$ *such that* P *fill the pillow diagram which is the same as (40) turned sideways.*

Here, M_2 is the square which will eventually be named $M_1 \star_V \alpha$ when we define an action of 2-cells on squares.

Remark 10. One can easily this condition is analogous to our filler condition (39) in a double category by turning the diagram (40) on its side. What the diagram says is that when we have a square with two bigons - the top one arbitrary and the bottom one the identity - there is another square M_2 (the back face of a pillow diagram) and a filler 2-morphism $P \in \mathbf{2Mor}$ which fills the diagram. If one imagines turning this diagram on its side and viewing it obliquely, one sees precisely (39), as a dimension has been suppressed. What is a square in (39) is a cylinder (2-morphism in **2Mor**); the roles of both squares and bigons in (40) are played by arrows in (39); arrows in (40) become pointlike objects in (39).

However, to get the compatibility between horizontal and vertical actions, we need something more than this. In particular, since these involve both horizontal and vertical cylinders (3-dimensional morphisms in the general sense), the compatibility condition must correspond to the 4-dimensional 2-cells in **2Mor**, shown in the lower right corner of Table 2.

To draw necessary condition is difficult, since the necessary diagram is four-dimensional, but we can describe it as follows:

Definition 12. *We say a double bicategory satisfies the* ***action compatibility condition*** *if the following holds. Suppose we are given*

- *a morphism* $F \in \mathbf{Mor}$
- *an object* $\alpha \in \mathbf{2Mor}$ *whose target in* $\mathbf{Mor}$ *is a source of* F
- *a 2-cell* $\beta \in \mathbf{Obj}$ *whose target morphism is a source of* F
- *an invertible morphism* $P_1 \in \mathbf{2Mor}$ *with* F *as source, and the objects* α *and* id *in* $\mathbf{2Mor}$ *as source and target*
- *an invertible 2-cell* $P_2 \in \mathbf{Mor}$ *with* F *as source, and the 2-cells* β *and* id *in* $\mathbf{Mor}$ *as source and target*

where P_1 *and* P_2 *have, as targets, morphisms in* $\mathbf{Mor}$ *we call* $\alpha \star F$ *and* $\beta \star F$ *respectively. Then there is a unique morphism* $\hat{F}$ *in* $\mathbf{Mor}$ *and 2-cell* T *in* $\mathbf{2Mor}$ *having all of the above as sources and targets.*

Geometrically, we can think of the unique 2-cell in $\mathbf{2Mor}$ as resembling the structure in the bottom right corner of Table 2. This can be seen as taking one horizontal cylinder to another in a way that fixes the (vertical) bigons on its sides, by means of a translation which acts on the front and back faces with a pair of vertical cylinders (which share the top and bottom bigonal faces). Alternatively, it can be seen as taking one vertical cylinder to another, acting on the faces with a pair of horizontal cylinders. In either case, the cylinders involved in the translation act on the faces, but the four-dimensional interior, T, acts on the original cylinder to give another. The simplest interpretation of this condition is that it is precisely the condition needed to give the compatibilty condition (29).

Remark 11. Notice that the two conditions given imply the existence of unique data of three different sorts in our double bicategory. If these are the only data of these kinds, we can effectively omit them (since it suffices to know information about their sources and targets. This omission is part of a decategorification of the same kind we saw for the double category $\mathbf{DC}$.

In particular, we use the above conditions to show the following:

Lemma 4. *Suppose* $\mathbf{IB}$ *is a double bicategorywhich has at most a unique morphism or 2-morphisms in* $\mathbf{2Mor}$*, and at most a unique 2-morphism in* $\mathbf{Mor}$*, having any specified sources and targets; and* $\mathbf{IB}$ *satisfies the horizontal and vertical action conditions and the action compatibility condition; then* $\mathbf{IB}$ *gives a Verity double bicategory in the sense of Verity.*

Proof. To begin with, we describe how the elements of a Verity double bicategory $\mathbf{DB}$ (definition 4) arise, given such an $\mathbf{IB}$, consisting of bicategories $(\mathbf{Obj}, \mathbf{Mor}, \mathbf{2Mor})$ together with all required maps (three kinds of source and target maps, two kinds of identity, three partially-defined compositions, left and right unitors, and the associator).

The horizontal bicategory $\mathbf{Hor}$ of $\mathbf{DB}$ is simply $\mathbf{Obj}$. The vertical bicategory $\mathbf{Ver}$ consists of the objects of each of $\mathbf{Obj}$, $\mathbf{Mor}$, and $\mathbf{2Mor}$, where the required source, target and composition maps for $\mathbf{Ver}$ are just the object maps from those for $\mathbf{IB}$, which are all functors. We next check that this is a bicategory.

The source and target maps for $\mathbf{Ver}$ satisfy all the usual rules for a bicategory since the corresponding functors in $\mathbf{IB}$ do. Similarly, the composition maps satisfy (5), (6) and (7) up to natural isomorphisms: they are just object maps of functors which satisfy corresponding conditions. We nextillustrate this for composition.

In **IB**, there is an associator 2-natural transformation. That is, a partially defined bifunctor $\alpha : \mathbf{Mor}^3 \rightarrow \mathbf{2Mor}$ satisfying the pentagon identity (strictly, since we are considering a *strict model* of the theory of bicategories). Among the data for α are the object maps, which give the maps for the associator in **Ver**. Since the associator 2-natural transformation satisfies the pentagon identity, so do these object maps. The other properties are shown similarly, so that **Ver** is a bicategory.

Next, we declare that the squares of **DB** are the morphisms of **Mor**. Their vertical source and target maps are the morphism maps from the source and target functors from **Mor** to **Obj**. Their horizontal source and target maps are the internal ones in **Mor**. These satisfy equations (21) because the source and target maps of **IB** are functors (in our special example of spans, this amounts to the fact that (35) commutes).

The horizontal composition of squares (23) is just the composition of morphisms in **Mor**. Now, by assumption, **Mor** is a bicategory with at most unique 2-morphisms having any given source and target. If we declare these are identities (that is, identify their source and target morphisms), we get that horizontal composition is exactly associative and has exact identities.

The vertical composition of squares (22) is given by the morphism maps for the partially defined functor $\circ$ for **Mor**, and so composition here satisfies the axioms for a bicategory. In particular, it has an associator and a unitor: but these must be morphisms in **2Mor** since we take the morphism maps from the associator and unitor functors (and the theory of bicatories says that these give 2-morphisms). But again, we can declare that there are only identity morphisms in **2Mor**, and this composition is exactly associative.

The interchange rule (24) follows again from functoriality of the composition functors.

The action of the 2-morphisms (bigons) on squares is guaranteed by the horizontal and vertical action conditions. In particular, by composition of in **Mor** or **2Mor**, we guarantee the existence of the categories of horizontal and vertical cylinders $\mathbf{Cyl_H}$ and $\mathbf{Cyl_V}$, respectively. These come from the 2-morphisms in **Mor** or morphisms in **2Mor** respectively which those conditions demand must exist. Taking these to be identities, the cylinders consist of commuting cylindrical diagrams with two bigons and two squares.

In the case where one bigon is the identity, and the other is any bigon α, the conditions guarantee the existence of a cylinder, which we have declared to be the identity. This defines the effect of the action of α on the square whose source is the target of α. If this square is F, we denote the other square $\alpha \star_H F$ or $\alpha \star_V F$ as appropriate.

The condition (27) guaranteeing independence of the horizontal and vertical actions follows from the action compatibility conditon. For suppose we have a square F whose horizontal and vertical source arrows are the targets of 2-cells α and β, and attach to its opposite faces two identity 2-cells. Then the horizontal and vertical action conditions mean that there will be a square $\alpha \star_H F$ and a square $\beta \star_V F$). Then the action compatibility condition applies (the P_i are the identities we get from the action condition), and there is a morphism in **Mor** - that is, a square in **DB** we can call and a 2-cell $T \in \mathbf{2Mor}$. Consider the remaining face, which the action condition suggests we call $\alpha \star_H (\beta \star_V F)$ or $\beta \star_V (\alpha \star_H F)$, depending on the order in which we apply them. The compatibility condition says that there is a unique square which fills this spot so the two must be equal.

So from any such double bicategory we get a Verity double bicategory.

□

Remark 12. It is interesting to note how these arguments apply to the case when we are looking at constructions in 2Span($\mathbf{C}$), as will be the case in $nCob$.

In particular, the interchange rules hold because the middle objects in the four squares being composed form the vertices of a new square. The pullbacks in the vertical and horizontal direction form the middle objects of vertical and horizontal spans over these. The interchange law means that the pullback (in the horizontal direction) of the objects from the vertical spans is in the same isomorphism class as the pullback (in the vertical direction) of the objects from the horizontal spans. This is true because of the universal property of the pullback.

The horizontal and vertical 2-morphisms are maps of spans, and act on the squares by composition of morphisms in $\mathbf{C}$: given a square M with four maps P_i and Π_i to the edges as in (35); and a morphism of spans on any edge (for definiteness, say the top), where the $\mathbf{C}$-morphism in the middle is $S \xrightarrow{f} \tilde{S}$. Then the composite $f \circ P_1 : M \to \tilde{S}$ is a source (or target) map to the span $X \xleftarrow{\pi_1} \tilde{S} \xrightarrow{\pi_2} Y$. The result is again a square. In particular, composition of internal maps in horizontal and vertical morphism of spans with the projections in a square are independent.

7. A Low Dimensional Example

Lauda and Pfeiffer [LP] describe an extended topological quantum field theory defined on "open-closed strings". These can be described in terms of the sort of cobordisms between cobordisms we have described in this paper. They describe a category of cobordisms in which the objects are compact one-dimensional manifolds, possibly with boundary: that is, either line segments, or circles. The morphisms joining these are generated by: all those for **2Cob** as shown in figure 2; an analogous set of generators with line segments instead of circles for objects; and generators passing from line segment to circle, and vice versa.

They describe the boundary edges as labelled by colourings, but in a fashion which is equivalent to identifying them as horizontal and vertical morphisms in a double (bi)category, as we have done here. In particular, the objects will be collections of zero or more points, the horizontal and vertical morphisms between such objects will be collections of circles or lines with endpoints in the sets. The 2-morphisms of these will be diffeomorphisms. The squares of the Verity double bicategorywill be the diffeomorphism classes of cobordisms with corners.

What ([LP], sec. 3.1.3), following [Laur] describes as a $\langle 2 \rangle$-diagram of inclusions in **Top**:

(41)
$$\begin{array}{ccc} \partial_0 M \cap \partial_1 M & \longrightarrow & \partial_0 M \\ \downarrow & & \downarrow \\ \partial_1 M & \longrightarrow & M \end{array}$$

can be recovered from a diagram of the form (35). This is done by taking each cospan in **Man** and replacing the two inclusions into the middle object by a single inclusion from the disjoint union of the source and target. This recovers the faces of the cobordism seen as a $\langle 2\rangle$-manifold.

So in particular, given the Verity double bicategory of cobordisms $\mathbf{2Cob_2}$ as described here, it is possible to recover the structure of the category $\mathbf{2Cob^{ext}}$ of open-closed cobordisms in the following way:

The objects of $\mathbf{2Cob^{ext}}$ are diffeomorphism classes of horizontal morphisms in $\mathbf{2Cob_2}$. A horizontal morphism in $\mathbf{2Cob_2}$ consists of a cobordism between two 0-manifolds X and Y. This is a 1-manifold S with boundary, $\partial S = X \coprod Y$: up to diffeomorphism (that is, horizontal 2-isomorphism in $frm{-}eCob_2$) this just amounts to a collection of circles and line segments. (One difference between our framework and that of Lauda and Pfeiffer is that they consider objects to be such collections with a definite order, by treating them as finite sequences with entries in $0, 1$, together with maps taking 0 to a circle, and 1 to a line segment, embedded in $\mathbb{R}^2$ in order along a line.)

One should note that the objects in $\mathbf{2Cob^{ext}}$ contain less information than the horizontal morphisms in $\mathbf{nCob}_2$. In particular, there are several ways to get a line segment as the diffeomorphism class of a cobordism between points, as shown in figure 5. The line segment can appear as a cobordism from two points to zero, or from zero to two, or from one to one.

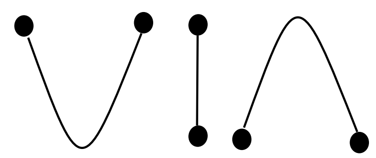


FIGURE 5. Three Cobordisms Diffeomorphic to a Line Segment

The category $\mathbf{2Cob^{ext}}$ makes no use of the vertical bicategory in $\mathbf{2Cob_2}$, but it has the same structure as the horizontal.

Section 4 of [LP] defines an open-closed TQFT as a symmetric monoidal functor from $\mathbf{2Cob^{ext}}$ into a symmetric monoidal category $\mathbf{C}$. It is a matter for future research to see how to develop such a construction in the case of the Verity double bicategory $\mathbf{nCob}_2$.

8. Conclusions and Further Directions

8.1. **Presentation of $\mathbf{nCob}_2$.** The key example in this paper has been $\mathbf{nCob}_2$, the Verity double bicategory of n-dimensional cobordisms between $(n-1)$-dimensional cobordisms between $(n-2)$-dimensional manifolds. This was seen as a generalization of $\mathbf{nCob}$, the category of n-dimensional cobordisms between $(n-1)$-dimensonal manifolds. In section 2.1 we recalled how to present the symmetric monoidal category $\mathbf{2Cob}$ as equivalent to the free such category on a collection generating objects and morphisms satisfying certain relations. That is, described a minimal, sufficient set of generators and relations for that category. This naturally raises the question of whether we can similarly present a minimal, sufficient set of generators and relations for $\mathbf{nCob}_2$.

To do this we would need generators for the objects, horizontal and vertical morphisms and 2-morphisms, and squares. This is a harder problem than we intend to deal with here, but we can make a few preliminary remarks. We have already described in section 7 how the category $\mathbf{2Cob^{ext}}$ of open-closed strings described by Lauda and Pfeiffer in [LP], extended only slightly, is just $\mathbf{2Cob_2}$. In proposition 3.10 of that paper, they discuss a presentation of that category in terms of generators which is readily extended to a presentation for $\mathbf{2Cob_2}$. The only difference is that one would need to describe a larger set of generators for a full Verity double bicategory.

The objects in $\mathbf{2Cob^{ext}}$ are (equivalence classes of) horizontal morphisms in $\mathbf{2Cob_2}$, so one needs in addition to describe the 2-morphisms as all diffeomorphisms of the open and closed "strings". Its morphisms become squares, and have the presentation described there. The vertical morphisms can be deduced as the boundaries of these.

In the case where $n = 3$, the problem of giving a presentation for $\mathbf{nCob}_2$ is significantly more difficult, although we can notice that the horizontal and vertical bicategories are just the extended form of **2Cob**, so the generators for objects and morphisms are already known. The 2-morphisms are all diffeomorphisms. It is also not too difficult to describe a set of generators for the squares by the use of Morse theory (and its generalization, Cerf theory) to find 3-dimensional cobordisms with corners having only one topology change. However, finding a necessary and sufficient set of relations for these is beyond the scope of this paper.

8.2. ***n*-tuple Biategories.** Describing a Verity double bicategory is a special case of describing a weak form of higher dimensional categories, or a *weak n-category.* This broader problem is discussed in more detail by Tom Leinster [Lei], and by Eugenia Cheng and Aaron Lauda [CL]. In light of this more general problem, we can suggest some directions in which to extend this concept further. One is to generalize the concept of a Verity double bicategory to a *n-tuple bicategory.*

We have seen how to construct $2\mathrm{Span}(\mathbf{C})$ for a general category $\mathbf{C}$ with limits (or $\mathbf{C}\,\mathrm{Cosp}^2$ for a $\mathbf{C}$ with colimits), and how we take a restricted form of this construction to yield a Verity double bicategory of cobordisms. We have chosen to stop the process of taking spans in a category of spans after two steps, but we could continue this construction. Taking spans in this new category gives cubes of objects with maps from corners to the middles of edges, from middles of edges to middles of faces, and from middlse of faces to the middle of the cube. Similarly, for any finite n, we can iterate the process of taking spans to yield an n-dimensional cube.

In particular, we note that "Verity double bicategories" are restricted cases of bicategories internal to **Bicat**. There is a category of all such structures, namely the functor category of all maps $F : Th(\mathbf{Bicat}) \to \mathbf{Bicat}$, denoted $[Th(\mathbf{Bicat}), \mathbf{Bicat}]$. There will be an analogous concept of "triple bicategories", namely bicategories internal to $[Th(\mathbf{Bicat}), \mathbf{Bicat}]$. In general, a "$k$-tuple bicategory" will be a bicategory internal to the category of weak $(k-1)$-tuple categories.

We conjecture here that for all k, a k-tuply iterated process of taking spans of spans (or cospons) will yield examples of these structure. If this is true, cobordisms with codimension k between objects and the highest-dimensional cobordism will naturally form a weak k-tuple category. To do this, one would naturally describe the cobordisms as $\langle k \rangle$-manifolds.

A further direction of generalization would be to substitute tricategories, tetracategories, and so forth in place of bicategories in the preceding construction, perhaps making different choices each stage. The question then arises what sort of structures it would be possible to define by selectively decategorifying, and what sorts of "filler" conditions this would need.

9. Acknowledgements

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Mathematics Department, University of California, Riverside
E-mail address: `morton@math.ucr.edu`