Feedback, trace and fixed point semantics

N. Sabadini, R.F.C. Walters
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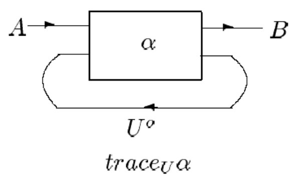
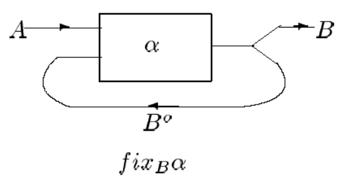
Aim

- Describe a new algebra of feedback
  - Related to trace monoidal categories of Joyal Street and Verity
  - Applies to feedback with delay
- Describe the free algebras
  - Categories of automata
  - Generalization of localization construction in ring theory

Algebras of feedback

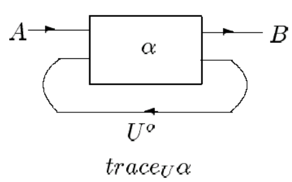
- Feedback, trace, fixed point operations
  - Have been studied by Elgot-Esik-Bloom, Stefanescu, Joyal-Street-Verity, Abramsky, Cockett, Plotkin, Hasegawa, …
  - In each case the basic algebra is a monoidal category ***A***, but different authors study in addition different operations:
    - *Trace: **A**(A1U,B1U) □**A**(A,B)*

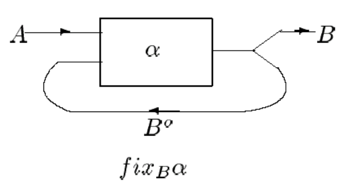




Compact closed categories

- are monoidal categories with two constants
  - *□U : I □U°1U and □U: U1U° □I*
  - satisfying (□U1 1U)(1U1 □U)=1U and (1U°1 □U)(□U11U°)=1U°.
  - represented geometrically by curved wires □ and
- can express trace and fixed point





$$trace_U(\alpha : A\otimes U \to B\otimes U) = (1_B \otimes \varepsilon_U)(\alpha\otimes 1_{U^\circ})(1_A \otimes \eta_{U^\circ}),$$

$$fix_B(\alpha : A\times B \to B) = (1_B \times \varepsilon_B)(\Delta_B \times 1_{B^\circ})(\alpha \times 1_{B^\circ})(1_A \times \eta_{B^\circ})$$

Analogy

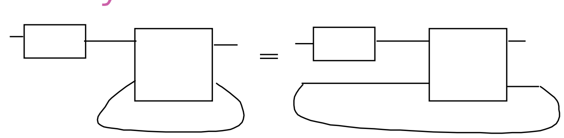
- categories with only identity arrows are sets
- monoidal cats with only identity arrows are monoids
- compact closed cats with only identity arrows are abelian groups

- There are classical (free) constructions
  - comm monoids → cancellative monoids → abelian group
  - the second of these is the construction of the integers from the natural numbers
- Joyal-Street-Verity
  - found a generalization of cancellative monoid (traced monoida category) and
  - generalized the integer construction

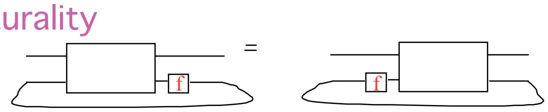
Traced monoidal categories

- Generalize cancellative monoids
- Axioms of trace (some only)

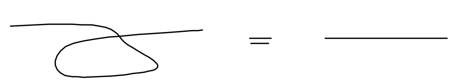
1. naturality



2. naturality



3. yanking



Feedback with delay

- The JSV axioms apply to feedback without delay
  - The two axioms which are false when the feedback has delay are the 2^{nd} naturality axiom and the yanking axiom.
  - However a weak form of the 2^{nd} naturality axiom does hold for feedback with delay _ the case in which the arrow f is an isomorphism.
  - In fact JSV prove that the strong 2^{nd} naturality axiom is implied by the weak version in the presence of the other axioms including yanking (not all given here).

DEFINITION A category with feedback (with delay) is a symmetric monoidal category satisfying the JSV axioms excluding yanking, and with ony the weak form of the 2^{nd} naturality axiom.

Categories with feedback

Note: a category with feedback in which

feedback(twist)=1

is a traced monoidal category.

This arrow feedback(twist) is the natural notion of delay in a category with feedback.

Why do we feel confident in our definition?

- There are a range of examples of computer science interest.
- The free construction is simple and interesting.
- It generalizes the classical comm monoid□ cancellative monoid construction

Free categories with feedback

THEOREM Given a symmetric monoidal category ***A*** the free category with feedback ***CIRC(A)*** on ***A*** is formed as follows:

- Objects of ***CIRC(A)*** are objects of ***A***.
- Arrows from *A* to *B* in ***CIRC(A)*** are pairs (□,*U*) such that □ is an arrow of ***A*** from *A* *U* to *B* *U*.
- Composition and tensor are given as follows (where the □' s are suitable permutations):

$$(\beta, V) \cdot (\alpha, U) = (\pi(\beta \otimes U)\pi(\alpha \otimes V), U \otimes V),$$

$$(\alpha, U) \otimes (\beta, V) = (\pi \cdot (\alpha \otimes \beta) \cdot \pi, U \otimes V).$$

Analogy

- The free cancellative monoid on a commutative monoid A consists of equivalence classes of elements of A:

  a□b iff there a.u=b.u for some u in A

- The free category with feedback on a symmetric monoidal category ***A*** consists of

  objects A, B,.. and arrows A U□ B U.

Examples

- *A* = (Sets, □). An arrow in *CIRC(A)* is a deterministic Mealy automaton. In composition the output actions of the first automaton is fed as input actions to the second. Tensor is parallel composition.
- *A* = (Sets, +). An arrow in *CIRC(A)* is a deterministic Elgot automaton _ a model of a sequential algorithm. In composition the final states of the first automaton become initial states of the second.
- *A* = ($Matr_{□□}$, ⊕). An arrow in *CIRC(A)* is a non deterministic automaton over the alphabet □.
- *A* = (Vect, ⊕). An arrow in *CIRC(A)* is a family of recursive linear equations (which may be thought of as defining either a continuous or discrete linear system).
- *A* = (Theory of rings, □). An arrow in *CIRC(A)* is a family of recursive polynomial equations. Composition is substitution of one system in the next.

Remarks

A further example like the last two

Take any algebra of computing processes, and consider *A* the algebraic theory, with tensor being product. Then an arrow in *CIRC(A)* is a recursive program _ ie a set of recursive equations in the algebra.

Fixed point semantics

The algebras *CIRC(A)* may be thought of as algebras of systems. Their fixed point semantics is often a feedback preserving functor from *CIRC(A)* to a compact closed category, which kills the delay.

More details available in
http://www.unico.it/~walters/papers/fics.pdf